

On a singular heat equation with dynamic boundary conditions

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February 21, 2013

Abstract

In this paper we analyze a singular heat equation of the form $\vartheta_t + \Delta\vartheta^{-1} = f$. The singular term ϑ^{-1} gives rise to very fast diffusion effects. The equation is settled in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ and complemented with a general *dynamic* boundary condition of the form $\alpha\vartheta_t - \beta\Delta_\Gamma\vartheta = \partial_n\vartheta^{-1}$, where Δ_Γ is the Laplace-Beltrami operator and α and β are nonnegative coefficients (in particular, the homogeneous Neumann problem given by $\alpha = \beta = 0$ is included). For this problem, we first introduce a suitable weak formulation and prove a related existence result. For more regular initial data, we show that there exists at least one weak solution satisfying instantaneous regularization effects which are uniform with respect to the time variable. This implies uniqueness for strictly positive times.

Key words: very fast diffusion, Moser iterations, dynamic boundary conditions.

AMS (MOS) subject classification: 35K55, 35K67, 35K51, 80A22.

1 Introduction

In this paper we address a singular heat equation having the expression

$$\vartheta_t + \Delta\vartheta^{-1} = f, \quad (1.1)$$

where f is an external force. The equation is settled in a smooth bounded domain $\Omega \subset \mathbb{R}^3$, where the restriction to the three-dimensional setting is motivated by physical applications, and is complemented with *dynamic* boundary conditions of the form

$$\alpha\vartheta_t - \beta\Delta_\Gamma\vartheta = \partial_n\vartheta^{-1} \quad \text{on } \Gamma := \partial\Omega, \quad (1.2)$$

where Δ_Γ is the Laplace-Beltrami operator and $\alpha, \beta \geq 0$ (in particular they may both be 0, so that the homogeneous Neumann problem is included). Equation (1.1) can be viewed in the framework of nonlinear diffusion problems, i.e., of equations of the form

$$\vartheta_t - \operatorname{div}(\vartheta^{m-1}\nabla\vartheta) = f, \quad (1.3)$$

with $m \in \mathbb{R}$. When $m \geq 1$, equation (1.3) is the well-known porous medium equation [18] (heat equation when $m = 1$); for $m < 1$, equation (1.3) lies in the class of fast diffusion equations (ultra fast when $m < 0$); actually, in this regime very fast diffusion occurs in the regions where ϑ is small. In the fast diffusion regime, one can also distinguish two sub-regimes. In this context, a crucial role is played by the so-called *first critical fast diffusion exponent* $m_c := \frac{d-2}{2}$ (d is the dimension) which acts as a threshold between the good parameter range $m \in (m_c, 1)$ and its complementary range $m \leq m_c$. More precisely, when $m_c < m < 1$ and the initial condition ϑ_0 is non-negative and locally integrable, then equation (1.3) (with $f = 0$) on the whole space has a unique weak solution which is globally defined, positive and smooth. Moreover, solutions emanating from initial data lying in $L^p(\mathbb{R}^d)$ ($p \geq 1$), or even in the Marcinkiewicz space $M^p(\mathbb{R}^d)$, immediately become bounded. The scenario when $m < m_c$ is drastically different for at least two reasons (we refer to [3] for further remarks). First of all, solutions may in general be unbounded and may also be non-smooth. As an example, when $\Omega = \mathbb{R}^3$, one can consider, for arbitrary $T > 0$, the function (see [17])

$$\Theta(t, x) = 2((T - t)^+ |x|^{-2})^{1/2}, \quad (1.4)$$

which solves (1.1) with $f = 0$ and initial condition $\Theta_0(x) = 2T^{1/2}|x|^{-1} \in L^p_{\text{loc}}(\mathbb{R}^3)$ for any $p < 3$. In particular, Θ is an unbounded function and $\Theta(t, \cdot) \notin L^p_{\text{loc}}(\mathbb{R}^3)$ for $p \geq 3$, at least until it vanishes for $t \geq T$. In this range of m , the boundedness of the solutions is tied to the summability of the initial condition and to the value of m . More precisely, solutions are bounded whenever $\vartheta_0 \in L^p(\mathbb{R}^d)$ with $p > p_c := \frac{d(1-m)}{2}$ (see [3]). Note that, when $d = 3$ and $m = -1$ (as in our case (1.1)), $p_c = 3$. A second notable feature of the fast diffusion regime, which is already evident in example (1.4), is the possible occurrence of extinction in finite time. This means that a solution may become identically zero after some finite time T which depends on the initial conditions. Consequently, positivity is lost. The ultra fast diffusion regime $m < 0$ presents further difficulties linked to the mere question of existence (see [16] and [11]). In particular, for (nonzero) data in L^p with $p < p_c$ one may face a phenomenon called “immediate extinction” meaning that solutions obtained as limits of reasonable approximation schemes can be identically zero for any $t > 0$. Note that the immediate extinction can occur also for boundary value problems with zero Dirichlet conditions. Finally, the case $m = m_* = \frac{d-4}{d-2}$, $d \geq 3$, deserves a particular attention, as observed in the papers [1] and [4] dealing with the asymptotics as $t \nearrow T$ (the extinction time). Note that we always have $m_* < m_c$ and, for $d = 3$, we have $m_* = -1$, exactly as in our equation (1.1).

In this paper, we are interested in the analysis of equation (1.1) with the dynamic boundary condition (1.2) in the case when the source term f has zero spatial mean. Then, a straightforward computation permits to see that conservation of mass occurs, namely $\int_{\Omega} \vartheta(t, x) dx = \int_{\Omega} \vartheta_0(x) dx$ for any $t > 0$, which manifestly excludes both immediate extinction and extinction in finite time. Our interest is twofold. From the one hand, we aim at proving existence of at least one solution under weak conditions on the initial data. To explain what we mean for “weak”, and considering for simplicity the case $f = 0$, we note that, thanks to the monotone structure of (1.1)-(1.2), the system admits several Liapunov functionals. In particular, physical considerations (see below for more details) suggest to introduce an *energy* functional \mathcal{E} (defined in (2.12) below). Mathematically, the finiteness of \mathcal{E} seems to be a minimal regularity condition on the initial datum ϑ_0 that allows to define a rigorous concept of weak solution (cf. Definition 2.1 below). Indeed (in the simpler Neumann case $\alpha = \beta = 0$), finiteness of the energy corresponds to asking that $\vartheta_0 - \log \vartheta_0 \in L^1(\Omega)$. Hence, we have some control on the L^1 -norm of the solution (permitting to use L^1 -arguments in the analysis) and also a positivity condition. Weak solutions emanating from initial data with finite energy \mathcal{E} will be called “energy” solutions. Actually, our first result (see Theorem 5.1 below) states that, if the initial data have finite energy and f has zero mean value and satisfies suitable summability conditions, then at least one global energy solution exists; moreover, the energy $\mathcal{E}(\vartheta)$ remains bounded uniformly in time.

Once existence is established, we study boundedness and positivity properties of energy solutions. Assuming the (sole) energy regularity of initial data, we can prove (see Theorem 5.1) that $\vartheta(t)$ becomes instantaneously bounded from below, namely, for $t > 0$ we have $\vartheta(t) \geq c(t) > 0$, where the constant $c(t)$ depends only on the initial energy and may vanish as $t \rightarrow 0$. If, in addition, ϑ_0 is (in space dimension $d = 3$) in some L^p -space with $p > 3 (= p_c)$, then (see Theorem 5.4) we prove an analogous bound from above ($\vartheta(t) \leq C(t) < +\infty$ for $t > 0$). These bounds also entail further

regularity properties of the solutions, which hold uniformly for large values of the time variable. Indeed, $c(t)$ does not vanish, and $C(t)$ does not explode, for $t \nearrow \infty$. The main tool in our regularity proof is a suitably modified Moser iteration scheme. Note that the required extra regularity of the initial datum is in complete agreement with the above discussion on the case $\Omega = \mathbb{R}^d$ and with the explicit solution (1.4). However, since the choice of dynamic boundary conditions (1.2) precludes the occurrence both of immediate and of finite time extinction, it remains an open challenging question to understand whether the restriction $p > p_c$ is still optimal in the present setting. Mathematically, the main difficulties we encountered in the analysis of this problem come from the choice of the boundary conditions (1.2) which does not allow us to perform some otherwise standard a priori estimates. Actually, we have a sort of “asymmetry” of diffusion effects between the equation in the interior domain (where the Laplacian acts on $1/\vartheta$) and that on the boundary (where the Laplace-Beltrami operator acts on (the trace of) ϑ). Moreover, as the precise statements of our results show, we cannot always allow α or β to be 0 (while they can be simultaneously be 0 since in that case we have the simpler case of no-flux conditions).

A natural application of equation (1.1) comes from the so-called phase change models of Penrose-Fife type [13]. In this physical context, the unknown ϑ represents the absolute temperature of a material liable to a phase transition, while the source f can also take into account the effects of the phase variable on the temperature evolution. More precisely, in the Penrose-Fife model, equation (1.1) is coupled with a parabolic equation of Allen-Cahn or Cahn-Hilliard type describing the evolution of the phase variable χ (see, e.g., [8], [9], [10], [14], [15]). In this framework, the choice of considering a zero-mean-valued forcing function f in (1.1) can be motivated by the need of replicating in our situation some inner cancellation effects that appear in the energy estimates for the full model. An application of the present results to the Penrose-Fife system with boundary conditions of the type (1.2) (which has never been studied in the literature, at least up to our knowledge) will be given in a forthcoming paper. Besides the Penrose-Fife model, equation (1.1) (or, more generally, equation (1.3) with $m < 0$) comes naturally into play in other physical contexts (see [2] and [7]). For example, it appears [12] in the study of the long-range Van der Waals interactions in thin films that diffuse on a solid surface.

The plan of the paper is as follows. In the next Section 2, we present our assumptions and state a rigorous definition of weak solution. Section 3 is devoted to proving the main technical lemmas related to the regularization properties of solutions: in particular, we show, by means of Moser iteration arguments, that, under suitable assumptions, both ϑ and ϑ^{-1} satisfy instantaneous regularization properties. Section 4 is devoted to showing (global) existence for smooth and bounded initial data. Finally, in Section 5 we prove our main results. These regard existence, regularization properties, and (in some cases) uniqueness of weak solutions.

2 Notation and hypotheses

Let Ω be a smooth bounded domain of \mathbb{R}^3 (of course, everything could be easily extended to the one and two dimensional cases, where, actually, better results are expected to hold). Let also $|\Omega| = 1$ so that $\|v\|_{L^p(\Omega)} \leq \|v\|_{L^q(\Omega)}$ for all $1 \leq p \leq q \leq +\infty$, $v \in L^q(\Omega)$. Let $H := L^2(\Omega)$, endowed with the standard scalar product (\cdot, \cdot) and norm $\|\cdot\|$. Let also $V := H^1(\Omega)$. We note by $\|\cdot\|_X$ the norm in the generic Banach space X . and by $\langle \cdot, \cdot \rangle_X$ the duality between X' and X . We will also write $\|\cdot\|_p$ for $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{p,\Gamma}$ for $\|\cdot\|_{L^p(\Gamma)}$, for simplicity. Next, we set $H_\Gamma := L^2(\Gamma)$ and $V_\Gamma := H^1(\Gamma)$ and denote by $(\cdot, \cdot)_\Gamma$ the scalar product in H_Γ , by $\|\cdot\|_\Gamma$ the corresponding norm, and by $\langle \cdot, \cdot \rangle_\Gamma$ the duality between V'_Γ and V_Γ . We also denote by ∇_Γ the tangential gradient on Γ and by Δ_Γ the Laplace-Beltrami operator. We can thus define the spaces

$$\mathcal{H} := H \times H_\Gamma \quad \text{and} \quad \mathcal{V} := \{z \in V : z|_\Gamma \in V_\Gamma\}. \quad (2.1)$$

Unless specified otherwise, in the sequel we shall make the following convention: when we write $h \in \mathcal{H}$, h will be interpreted as a pair of functions belonging, respectively, to H and to H_Γ , and both denoted by the same letter. On the other hand, when we consider $v \in \mathcal{V}$ (or even $v \in V$), the symbol v will be intended, depending on the context, either as a function defined on Ω , or as a pair formed by a function of Ω and its trace on Γ .

For any function, or functional z , defined on Ω , we can then set

$$m_\Omega(z) := \frac{1}{|\Omega|} \int_\Omega z = \int_\Omega z, \quad (2.2)$$

where the integral is substituted with the duality $\langle z, 1 \rangle$ in case, e.g., $z \in V'$. Given $\alpha \geq 0$, we also define the measure dm , given by

$$\int_\Omega v \, dm := \int_\Omega v + \alpha \int_\Gamma v, \quad (2.3)$$

where v represents a generic function in $L^1(\Omega) \times L^1(\Gamma)$. Here and below, integrals over Γ are to be intended with respect to the standard surface measure. With some abuse of notation, we will also write

$$m(v) := \frac{1}{|\Omega| + \alpha|\Gamma|} \int_\Omega v \, dm = \frac{1}{1 + \alpha|\Gamma|} \int_\Omega v \, dm, \quad (2.4)$$

i.e., the “mean value” of v w.r.t. the measure dm . Here $|\Gamma|$ represents the surface measure of Γ . In case $\alpha = 0$, it is intended that $m(v) = m_\Omega(v)$. For $p \in [1, \infty)$ and X a Banach space, we introduce the space

$$\mathcal{T}^p(0, +\infty; X) := \left\{ g \in L_{\text{loc}}^p(0, +\infty; X) : \sup_{t \in [0, +\infty)} \|g\|_{L^p(t, t+1; X)} < +\infty \right\}, \quad (2.5)$$

which is a Banach space as it is endowed with the graph norm. Assuming $f : (0, +\infty) \times \Omega \rightarrow \mathbb{R}$ be a suitable source term and letting $\alpha, \beta \geq 0$, we can introduce the system

$$\vartheta_t - \Delta u = f, \quad u = -\frac{1}{\vartheta}, \quad \text{in } (0, +\infty) \times \Omega, \quad (2.6)$$

$$\alpha \eta_t - \beta \Delta_\Gamma \eta = -\partial_n u, \quad \text{on } (0, +\infty) \times \Gamma, \quad (2.7)$$

$$\vartheta|_{t=0} = \vartheta_0 \quad \text{in } \Omega, \quad (2.8)$$

$$\alpha \eta|_{t=0} = \alpha \eta_0 \quad \text{on } \Gamma. \quad (2.9)$$

Our basic assumptions on the initial data are the following:

$$\vartheta_0 \in L^1(\Omega), \quad \log \vartheta_0 \in L^1(\Omega), \quad (2.10)$$

$$\alpha \eta_0 \in L^1(\Gamma), \quad \alpha \log \eta_0 \in L^1(\Gamma). \quad (2.11)$$

This very natural condition corresponds to asking that the initial data have *finite energy*, where the energy functional \mathcal{E} is defined as

$$\mathcal{E}(\vartheta, \eta) := \int_\Omega (\vartheta - \log \vartheta) + \alpha \int_\Gamma (\eta - \log \eta). \quad (2.12)$$

In the sequel, we will simply write, with some abuse of language, $\mathcal{E}(t)$, in place of $\mathcal{E}(\vartheta(t), \eta(t))$. The source term is assumed to satisfy for some given $\epsilon \in (0, 1)$

$$f \in L^2(0, +\infty; L^{6/5}(\Omega)) \cap \mathcal{T}^2(0, +\infty; L^{3+\epsilon}(\Omega)), \quad m_\Omega(f)(t) = 0, \quad \text{for a.a. } t \in (0, +\infty). \quad (2.13)$$

The reason for taking a source term with zero-mean value is that we need to have a control of the spatial average both of ϑ and of u (and for general f it is not completely clear how to achieve the latter). The above assumptions suffice to define a suitable concept of weak solution:

Definition 2.1. A (global) weak solution (or “energy solution”) to Problem (P) is a triplet (ϑ, η, u) satisfying, for all $T > 0$, the regularity properties

$$\vartheta \in C^0([0, T]; L^1(\Omega)), \quad \alpha \eta \in C^0([0, T]; L^1(\Gamma)), \quad (2.14)$$

$$u \in L^2(0, T; V), \quad u = -1/\vartheta, \quad \text{a.e. in } \Omega, \quad (2.15)$$

$$\beta \log \eta \in L^2(0, T; V_\Gamma), \quad (2.16)$$

$$\eta = -1/u_\Gamma, \quad \text{a.e. on } (0, T) \times \Gamma, \quad (2.17)$$

and fulfilling, for any test function

$$\xi \in C^1([0, T]; C^0(\bar{\Omega})) \cap C^0([0, T]; C^2(\bar{\Omega})) \quad (2.18)$$

and for all times $t \in [0, T]$, the relation

$$\begin{aligned} & \int_{\Omega} \vartheta(t) \xi(t) + \alpha \int_{\Gamma} \eta(t) \xi(t) + \int_0^t \int_{\Omega} \nabla u \cdot \nabla \xi - \beta \int_0^t \int_{\Gamma} \eta \Delta_{\Gamma} \xi \\ &= \int_0^t \int_{\Omega} \vartheta \xi_t + \alpha \int_0^t \int_{\Gamma} \eta \xi_t + \int_0^t \int_{\Omega} f \xi + \int_{\Omega} \vartheta_0 \xi(0) + \alpha \int_{\Gamma} \eta_0 \xi(0). \end{aligned} \quad (2.19)$$

Remark 2.2. It is worth giving some explanation of relation (2.17). There, u_{Γ} denotes the trace of u on Γ , which exists for almost every value of the time variable thanks to the first (2.15). More precisely, we have that $u_{\Gamma} \in L^2(0, T; H^{1/2}(\Gamma))$. Actually, for weak solutions we cannot simply write $\eta = \vartheta_{\Gamma}$ since the trace of ϑ does not necessarily exist. On the other hand, when considering smoother solutions (e.g., in the approximation detailed in Section 4 below) it will happen that ϑ is more regular so that η can be intended, in fact, as the trace of ϑ .

3 Main technical lemmas

In this section we prove some regularization estimates holding for sufficiently smooth solutions of Problem (P). In this procedure, a sufficient regularity will always be assumed in such a way that all computations we perform make sense. For this reason, we will indicate simply by ϑ , rather than by η , the boundary temperature (indeed, the “bulk” ϑ has a trace at this regularity level). As a rule, we will use the letters c and κ to denote generic positive constants, depending only on the set Ω and on the parameters α and β , with κ used in estimates from below. The values of c , κ are allowed to vary on occurrence. Finally, Q will denote a generic computable function, increasingly monotone with respect to each of its parameters, taking values in $[0, +\infty)$. Neither the constants κ , c , nor the functions Q , are allowed to have an explicit dependence on time.

3.1 Regularization of u

Lemma 3.1 (Moser iterations for u). *Let $S \in \mathbb{R}$, $T > S$, $\alpha \geq 0$, $\beta \geq 0$, and let, for some $\epsilon \in (0, 1)$,*

$$f \in L^2(S, T; L^{3+\epsilon}(\Omega)), \quad \|f\|_{L^2(S, T; L^{3+\epsilon}(\Omega))} =: F. \quad (3.1)$$

Let (u, ϑ) be a couple of sufficiently smooth functions solving, in a suitable sense, the system

$$\vartheta_t - \Delta u = f, \quad \vartheta = -1/u, \quad \text{in } \Omega, \quad (3.2)$$

$$\alpha \vartheta_t - \beta \Delta_{\Gamma} \vartheta = -\partial_{\mathbf{n}} u, \quad \text{on } \Gamma, \quad (3.3)$$

over the time interval (S, T) . Moreover, let us assume that

$$u \in L^2(S, T; V), \quad \|u\|_{L^2(S, T; V)} =: M, \quad (3.4)$$

$$u(S) \in L^1(\Omega), \quad \alpha u(S) \in L^1(\Gamma), \quad \|u(S)\|_{L^1(\Omega)} + \alpha \|u(S)\|_{L^1(\Gamma)} =: U. \quad (3.5)$$

Then, for any $\tau \in (0, \min\{1, T - S\})$, we have

$$\|u\|_{L^{\infty}((S+\tau, T) \times \Omega)} \leq Q(F, M, U, \tau^{-1}). \quad (3.6)$$

PROOF. We will just consider the case when $\beta = 0$, which is more difficult since we cannot get any help from the boundary diffusion term. Setting $z := -u \geq 0$, we will also assume, for the sake of simplicity, that $z \geq 1$ almost everywhere. Indeed, if that does not hold, then it is easy to check that in the estimates below we can simply replace z with $z = \max\{z, 1\}$. That said, we test (3.2) by z^{p+1} , where $p \geq 1$ will be specified later. This gives

$$\frac{1}{p} \frac{d}{dt} (\|z\|_p^p + \alpha \|z\|_{p, \Gamma}^p) + \frac{4(p+1)}{(p+2)^2} \|\nabla z^{\frac{p+2}{2}}\|^2 \leq \int_{\Omega} |f| z^{p+1}. \quad (3.7)$$

In order to recover the full V -norm on the left hand side, we can multiply (3.7) by p and then add to both hands sides the quantity

$$\|z^{\frac{p+2}{2}}\|_1^2 = \|z\|_{\frac{p+2}{2}}^{p+2}. \quad (3.8)$$

Hence, using continuity of the embedding $V \subset L^6(\Omega)$ and of the trace operator from V to $L^4(\Gamma)$, we get on the left hand side of (3.7) a quantity \mathcal{I} such that

$$\mathcal{I} := \kappa \|\nabla z^{\frac{p+2}{2}}\|^2 + \|z\|_{\frac{p+2}{2}}^{p+2} \geq \kappa \|z\|_{3(p+2)}^{p+2} + \kappa \|z\|_{2(p+2), \Gamma}^{p+2}. \quad (3.9)$$

We now choose $p = 1$ for the first iteration. Then, we can estimate the right hand side of (3.8) as

$$\|z\|_{\frac{p+2}{2}}^{p+2} \stackrel{p=1}{=} \|z\|_{\frac{3}{2}}^3 \leq c \|z\|_6^2 \|z\|_1. \quad (3.10)$$

On the other hand, still for $p = 1$, we can write

$$\int_{\Omega} |f| z^2 \leq \|f\|_3 \|z\|^{1/2} \|z^{3/2}\|_6 \leq \sigma \|z\|_9^3 + c_{\sigma} \|f\|_3^2 \|z\|_1, \quad (3.11)$$

whence, taking σ small enough and collecting (3.8)-(3.11), (3.7) becomes

$$\frac{d}{dt} (\|z\|_1 + \alpha \|z\|_{1, \Gamma}) + \kappa \|z\|_9^3 + \kappa \|z\|_{6, \Gamma}^3 \leq c (\|z\|_6^2 + \|f\|_3^2) \|z\|_1. \quad (3.12)$$

Thus, integrating over (S, T) and using Gronwall's lemma, we readily arrive at

$$\|z\|_{L^{\infty}(S, T; L^1(\Omega))} + \alpha \|z\|_{L^{\infty}(S, T; L^1(\Gamma))} + \|z\|_{L^3(S, T; L^9(\Omega))} + \|z\|_{L^3(S, T; L^6(\Gamma))} \leq Q(F, M, U), \quad (3.13)$$

where we can notice that the function Q has no explicit dependence on T . This relation is the starting point for the subsequent iterations.

To proceed, we set $r := \frac{3+\epsilon}{2+\epsilon}$ to be the conjugate exponent of $3+\epsilon$. Rewriting (3.7) (multiplied by p) for a suitable new choice of p , we can now estimate the right hand side as follows:

$$p \int_{\Omega} |f| z^{p+1} \leq p \|f\|_{3+\epsilon} \|z^{p+1}\|_r = p \|f\|_{3+\epsilon} \|z\|_{r(p+1)}^{p+1}. \quad (3.14)$$

Then, adding again the term $\|z\|_{(p+2)/2}^{p+2}$ to both hand sides and integrating the result from some $t \geq S$ (to be chosen later) to T , we arrive at

$$\begin{aligned} & \|z\|_{L^{\infty}(t, T; L^p(\Omega))}^p + \alpha \|z\|_{L^{\infty}(t, T; L^p(\Gamma))}^p + \|z\|_{L^{p+2}(t, T; L^{3(p+2)}(\Omega))}^{p+2} + \|z\|_{L^{p+2}(t, T; L^{2(p+2)}(\Gamma))}^{p+2} \\ & \leq c \|z(t)\|_p^p + c\alpha \|z(t)\|_{p, \Gamma}^p + cpF \|z\|_{L^{2(p+1)}(t, T; L^{r(p+1)}(\Omega))}^{p+1} + c \|z\|_{L^{p+2}(t, T; L^{\frac{p+2}{2}}(\Omega))}^{p+2} \\ & \leq c \|z(t)\|_p^p + c\alpha \|z(t)\|_{p, \Gamma}^p + c(pF + 1) \|z\|_{L^{2(p+1)}(t, T; L^{r(p+1)}(\Omega))}^{p+2}, \end{aligned} \quad (3.15)$$

where we used, in particular, that $z \geq 1$. Next, we define, for $i \geq 0$ and τ_i to be chosen later,

$$J_i^{p_i} := \|z\|_{L^{\infty}(\tau_i, T; L^{p_i}(\Omega))}^{p_i} + \|z\|_{L^{p_i+2}(\tau_i, T; L^{3p_i+6}(\Omega))}^{p_i} + \|z\|_{L^{p_i+2}(t, T; L^{2(p_i+2)}(\Gamma))}^{p_i}. \quad (3.16)$$

Then, elementary interpolation gives

$$\begin{aligned} \|z\|_{L^{2(p_i+1+1)}(\tau_i, T; L^{r(p_i+1+1)}(\Omega))} & \leq \|z\|_{L^{\infty}(\tau_i, T; L^{p_i}(\Omega))}^{\rho} \|z\|_{L^{p_i+2}(\tau_i, T; L^{3p_i+6}(\Omega))}^{1-\rho} \\ & \leq \rho \|z\|_{L^{\infty}(\tau_i, T; L^{p_i}(\Omega))} + (1-\rho) \|z\|_{L^{p_i+2}(\tau_i, T; L^{3p_i+6}(\Omega))}, \end{aligned} \quad (3.17)$$

where the index p_{i+1} and the interpolation exponent $\rho = \rho(i)$ are given by the system

$$\begin{cases} \frac{1-\rho}{p_i+2} = \frac{1}{2(p_{i+1}+1)}, \\ \frac{\rho}{p_i} + \frac{1-\rho}{3(p_i+2)} = \frac{1}{r(p_{i+1}+1)}. \end{cases} \quad (3.18)$$

Dividing the second equation in (3.18) by the first one, we have

$$\left(\frac{\rho}{p_i} + \frac{1-\rho}{3(p_i+2)}\right) \frac{p_i+2}{1-\rho} = \frac{2}{r}, \quad (3.19)$$

whence

$$\frac{\rho}{p_i} \frac{p_i+2}{1-\rho} = \frac{2}{r} - \frac{1}{3} =: K_\epsilon, \quad (3.20)$$

and it is easy to compute

$$K_\epsilon = \frac{9+5\epsilon}{9+3\epsilon}. \quad (3.21)$$

From (3.20) we also infer

$$\rho = \frac{\frac{p_i}{p_i+2} K_\epsilon}{1 + \frac{p_i}{p_i+2} K_\epsilon} \in (0, 1), \quad \text{provided that } p_i \geq 1. \quad (3.22)$$

Being

$$1 - \rho = \frac{1}{1 + \frac{p_i}{p_i+2} K_\epsilon}, \quad (3.23)$$

we then obtain from the first (3.18)

$$\begin{aligned} p_{i+1} &= \frac{1}{2} \frac{p_i+2}{1-\rho} - 1 = \frac{1}{2} \left(1 + \frac{p_i}{p_i+2} K_\epsilon\right) (p_i+2) - 1 \\ &= \frac{K_\epsilon+1}{2} p_i = \frac{9+4\epsilon}{9+3\epsilon} p_i =: H p_i, \end{aligned} \quad (3.24)$$

where, obviously, $H = H(\epsilon) > 1$ whenever $\epsilon > 0$.

Taking the p_i -th power of (3.17) and using Young's inequality it is not difficult to infer

$$J_i \geq \|z\|_{L^{2(p_i+1+1)}(\tau_i, T; L^{r(p_i+1+1)}(\Omega))}. \quad (3.25)$$

We can now start the iteration argument. Let $p_0 = 1$ and inductively define, for $i \geq 1$, $p_{i+1} := H p_i = H^{i+1}$. Moreover, let (for instance), for $i \geq 1$,

$$\mathfrak{t}_i := \tau \frac{3}{\pi^2 i^2}, \quad \text{so that } \sum_{i=1}^{\infty} \mathfrak{t}_i = \frac{\tau}{2}. \quad (3.26)$$

Let us now rewrite (3.15) by taking $p = p_{i+1}$. Setting also, for brevity, $J_i := J_{p_i}$ and writing τ_{i+1} in place of \mathfrak{t} (to be chosen below), we then obtain, thanks also to (3.25),

$$J_{i+1}^{p_{i+1}} + \alpha \|z\|_{L^\infty(\tau_{i+1}, T; L^{p_{i+1}}(\Gamma))}^{p_{i+1}} \leq c \|z(\tau_{i+1})\|_{p_{i+1}}^{p_{i+1}} + c \alpha \|z(\tau_{i+1})\|_{p_{i+1}, \Gamma}^{p_{i+1}} + c(p_{i+1} F + 1) J_i^{p_{i+1}+2}. \quad (3.27)$$

Let us now make precise the choice of τ_i . By induction, given τ_i , we take $\tau_{i+1} \in (\tau_i, \tau_i + \mathfrak{t}_{i+1})$ such that

$$\begin{aligned} \|z(\tau_{i+1})\|_{p_{i+1}}^{p_{i+1}} + \alpha \|z(\tau_{i+1})\|_{p_{i+1}, \Gamma}^{p_{i+1}} &\leq \frac{1}{\mathfrak{t}_{i+1}} \int_{\tau_i}^{\tau_i + \mathfrak{t}_{i+1}} \left(\|z(t)\|_{p_{i+1}}^{p_{i+1}} + \alpha \|z(t)\|_{p_{i+1}, \Gamma}^{p_{i+1}} \right) dt \\ &\leq c \frac{i^2}{\tau} \int_{\tau_i}^{\tau_i + \mathfrak{t}_{i+1}} \left(\|z(t)\|_{3(p_i+2)}^{p_{i+1}} + \alpha \|z(t)\|_{2(p_i+2), \Gamma}^{p_{i+1}} \right) dt \leq c \frac{i^2}{\tau} J_i^{p_{i+1}}, \end{aligned} \quad (3.28)$$

where we used that $p_{i+1} \leq 2(p_i+2)$. Thus, (3.27)-(3.28) give

$$J_{i+1}^{p_{i+1}} + \alpha \|z\|_{L^\infty(\tau_{i+1}, T; L^{p_{i+1}}(\Gamma))}^{p_{i+1}} \leq c i^{2H} \tau^{-H} J_i^{p_{i+1}} + c(p_{i+1} F + 1) J_i^{p_{i+1}+2}. \quad (3.29)$$

In particular, we obtain

$$J_{i+1}^{H^{i+1}} \leq c \left(i^{2H} \tau^{-H} + H^{i+1} F + 1 \right) J_i^{H^{i+1}+2}. \quad (3.30)$$

Thus, setting $\eta_i := (H^i + 2)/H^i$, we have

$$J_{i+1} \leq B_i^{H^{-(i+1)}} J_i^{\eta_{i+1}}, \quad \text{where } B_i := c(i^{2H} \tau^{-H} + H^{i+1} F + 1). \quad (3.31)$$

Hence, letting $L_i := \log J_i$ and $\zeta_i := \log B_i$, we can rewrite (3.31), for i large enough, as

$$\begin{aligned} L_{i+1} &\leq H^{-(i+1)} \zeta_i + \eta_{i+1} L_i \leq H^{-(i+1)} \zeta_i + \eta_{i+1} (H^{-i} \zeta_{i-1} + \eta_i L_{i-1}) \\ &\leq H^{-(i+1)} \zeta_i + \eta_{i+1} H^{-i} \zeta_{i-1} + \eta_{i+1} \eta_i (H^{-(i-1)} \zeta_{i-2} + \eta_{i-1} L_{i-2}) \\ &\leq \cdots \leq L_0 \prod_{k=1}^{i+1} \eta_k + \sum_{k=1}^{i+1} \left(\zeta_{k-1} H^{-k} \prod_{j=k+1}^{i+1} \eta_j \right), \end{aligned} \quad (3.32)$$

where the last product is understood to be 1 for $k = i + 1$. A direct check then permits to obtain (3.6) (see also [14]). \blacksquare

If u is bounded at the initial time, we can avoid all complications connected with the choice of the sequence t_i . Indeed, a straightforward modification of the above proof permits to show the following

Corollary 3.2. *Let $S \in \mathbb{R}$, $T > S$, $\alpha \geq 0$, $\beta \geq 0$, and let, for some $\epsilon \in (0, 1)$, (3.1) hold. Let, as before, (u, ϑ) be a couple of sufficiently smooth functions solving, in a suitable sense, system (3.2)-(3.3) over the time interval (S, T) . Moreover, let us assume (3.4) together with*

$$u(S) \in L^\infty(\Omega), \quad \alpha u(S) \in L^\infty(\Gamma), \quad \|u(S)\|_{L^\infty(\Omega)} + \alpha \|u(S)\|_{L^\infty(\Gamma)} =: U. \quad (3.33)$$

Then, we have that

$$\|u\|_{L^\infty((S,T) \times \Omega)} \leq Q(F, M, U). \quad (3.34)$$

3.2 Regularization of ϑ

Lemma 3.3 (conservation of L^p norm). *Let $S \in \mathbb{R}$, $T > S$, $\alpha \geq 0$, $\beta \geq 0$. Let (u, ϑ) be a couple of sufficiently smooth functions solving in a suitable sense system (3.2)-(3.3) over (S, T) . Let also, for some $p \geq 1$,*

$$f \in L^2(S, T; L^3(\Omega)), \quad \|f\|_{L^2(S,T;L^3(\Omega))} =: F, \quad (3.35)$$

$$\vartheta \in L^\infty(S, T; L^1(\Omega)), \quad \|\vartheta\|_{L^\infty(S,T;L^1(\Omega))} =: L, \quad (3.36)$$

$$\vartheta(S) \in L^p(\Omega), \quad \alpha \vartheta(S) \in L^p(\Gamma), \quad \|\vartheta(S)\|_{L^p(\Omega)} + \alpha \|\vartheta(S)\|_{L^p(\Gamma)} =: \Theta. \quad (3.37)$$

Then,

$$\|\vartheta\|_{L^\infty(S,T;L^p(\Omega))} + \alpha \|\vartheta\|_{L^\infty(S,T;L^p(\Gamma))} \leq Q(L, F, \Theta). \quad (3.38)$$

PROOF. We give the proof only for $p > 2$. Actually, the case $p \in [1, 2]$ works similarly but requires some change of notation. Then, let us consider first the case when $p \leq 4$. We test (3.2) by ϑ^{p-1} . This gives

$$\frac{d}{dt} \left(\|\vartheta\|_p^p + \alpha \|\vartheta\|_{p,\Gamma}^p \right) + \frac{4p(p-1)}{(p-2)^2} \|\nabla \vartheta^{\frac{p-2}{2}}\|^2 + \beta \frac{4p(p-1)}{p^2} \|\nabla_\Gamma \vartheta^{\frac{p}{2}}\|_\Gamma^2 \leq p \int_\Omega |f| \vartheta^{p-1}. \quad (3.39)$$

Then, being $p \leq 4$, thanks to (3.36), we can add the inequality

$$\|\vartheta^{\frac{p-2}{2}}\|_1^2 \leq Q(L), \quad (3.40)$$

which permits to get the full V -norm of $\vartheta^{\frac{p-2}{2}}$ from the gradient term. Then, we estimate the right hand side of (3.39) as follows:

$$p \int_\Omega |f| \vartheta^{p-1} \leq c \|f\|_3 \|\vartheta^{\frac{p}{2}}\|_2 \|\vartheta^{\frac{p-2}{2}}\|_6 \leq \sigma \|\vartheta\|_{3(p-2)}^{p-2} + c_\sigma \|f\|_3^2 \|\vartheta\|_p^p. \quad (3.41)$$

Taking σ small enough, we arrive at the inequality

$$\frac{d}{dt}(\|\vartheta\|_p^p + \alpha\|\vartheta\|_{p,\Gamma}^p) + \kappa\|\vartheta\|_{3(p-2)}^{p-2} \leq c\|f\|_3^2\|\vartheta\|_p^p + Q(L), \quad (3.42)$$

whence the thesis follows by integrating in time and using (3.35), (3.37) and Gronwall's lemma. In the case $p > 4$, we cannot proceed directly in this way since (3.36) does not entail (3.40). However, it is clear that we can first take $p = 4$ to get an estimate of ϑ in $L^\infty(S, T; L^4(\Omega))$ (and of $\alpha\vartheta$ in $L^\infty(S, T; L^4(\Gamma))$) and then iterate the procedure finitely many times. \blacksquare

Lemma 3.4 (Moser iterations for ϑ). *Let $S \in \mathbb{R}$, $T > S$, $\alpha \geq 0$, $\beta \geq 0$. Moreover, if $\beta = 0$ let also $\alpha = 0$. Let (u, ϑ) be a couple of sufficiently smooth functions solving in a suitable sense system (3.2)-(3.3) over (S, T) . Moreover, let us assume that, for some $\epsilon \in (0, 1)$,*

$$f \in L^2(S, T; L^{3+\epsilon}(\Omega)), \quad \|f\|_{L^2(S, T; L^{3+\epsilon}(\Omega))} =: F, \quad (3.43)$$

$$\vartheta \in L^\infty(S, T; L^1(\Omega)), \quad \|\vartheta\|_{L^\infty(S, T; L^1(\Omega))} =: L, \quad (3.44)$$

$$\vartheta(S) \in L^{3+\epsilon}(\Omega), \quad \alpha\vartheta(S) \in L^{3+\epsilon}(\Gamma), \quad \|\vartheta(S)\|_{L^{3+\epsilon}(\Omega)} + \alpha\|\vartheta(S)\|_{L^{3+\epsilon}(\Gamma)} =: \Theta. \quad (3.45)$$

Then, for any $\tau \in (0, \min\{1, T - S\})$, we have

$$\|\vartheta\|_{L^\infty((S+\tau, T) \times \Omega)} + \alpha\|\vartheta\|_{L^\infty((S+\tau, T) \times \Gamma)} \leq Q(F, L, \Theta, \tau^{-1}). \quad (3.46)$$

PROOF. To start with a further Moser iteration procedure, we rewrite (3.39), multiply it by p , and then add to both sides the term $\|\vartheta^{\frac{p-2}{2}}\|_1^2$. This procedure gives

$$\frac{d}{dt}(\|\vartheta\|_p^p + \alpha\|\vartheta\|_{p,\Gamma}^p) + \kappa\|\vartheta\|_{3(p-2)}^{p-2} + \beta\kappa\|\vartheta\|_{3(p-2),\Gamma}^{p-2} \leq p \int_\Omega |f|\vartheta^{p-1} + c\|\vartheta\|_{\frac{p-2}{2}}^{p-2}. \quad (3.47)$$

Then, in the case $\alpha = \beta = 0$, the proof works similarly with [15, Lemma 3.5], to which we refer the reader for details. In the cases $\alpha, \beta > 0$ and $\alpha = 0, \beta > 0$, the argument can be adapted just with small variants. Note in particular that, at the first iteration step, we choose $p = p_0 = 3 + \epsilon$ so that the latter term in (3.47) can be controlled by using Lemma 3.3. \blacksquare

In the case when $\beta = 0$ and $\alpha > 0$, we can still prove regularization of ϑ , but the argument is a bit more delicate since we cannot take advantage of the boundary gradient term. So, we have to use the trace theorem as in Lemma 3.1. This, however, forces us to assume some more summability of the initial value of ϑ . Namely, we have

Lemma 3.5 (Moser iterations for ϑ , case $\alpha > 0, \beta = 0$). *Let $S \in \mathbb{R}$, $T > S$, $\alpha > 0$, and $\beta = 0$. Let (u, ϑ) be a couple of sufficiently smooth functions solving in a suitable sense system (3.2)-(3.3) over (S, T) . Moreover, let us assume that, for some $\epsilon \in (0, 1)$, (3.43)-(3.44) hold, together with*

$$\vartheta(S) \in L^{4+\epsilon}(\Omega), \quad \vartheta(S) \in L^{4+\epsilon}(\Gamma), \quad \|\vartheta(S)\|_{L^{4+\epsilon}(\Omega)} + \|\vartheta(S)\|_{L^{4+\epsilon}(\Gamma)} =: \Theta. \quad (3.48)$$

Then, for any $\tau \in (0, \min\{1, T - S\})$, (3.46) holds.

PROOF. We take for simplicity $\alpha = 1$. The analogue of (3.47) reads now

$$\frac{d}{dt}(\|\vartheta\|_p^p + \|\vartheta\|_{p,\Gamma}^p) + \kappa\|\vartheta^{\frac{p-2}{2}}\|_V^2 \leq p \int_\Omega |f|\vartheta^{p-1} + c\|\vartheta\|_{\frac{p-2}{2}}^{p-2}. \quad (3.49)$$

Then, we take first $p = 4 + \epsilon$. Owing to Lemma 3.3, we then get

$$\|\vartheta\|_{L^\infty(S, T; L^{4+\epsilon}(\Omega))} + \|\vartheta\|_{L^\infty(S, T; L^{4+\epsilon}(\Gamma))} \leq Q(F, L, \Theta). \quad (3.50)$$

Let us now repeat (3.49) for a new p to be chosen below. Let also $r = (3 + \epsilon')/(2 + \epsilon')$, where ϵ' is a number, also to be chosen below, in the range $(0, \epsilon]$. We then obtain

$$p \int_\Omega |f|\vartheta^{p-1} \leq p\|f\|_{3+\epsilon'}\|\vartheta^{p-1}\|_r \leq p\|f\|_{3+\epsilon}\|\vartheta\|_{r(p-1)}^{p-1}. \quad (3.51)$$

Hence, integrating (3.49) over (t, T) , for some $t \geq S$ (to be chosen later) and using once more continuity of the trace from V to $L^4(\Gamma)$, we arrive (compare with (3.15)) at

$$\begin{aligned} & \|\vartheta\|_{L^\infty(t, T; L^p(\Omega))}^p + \|\vartheta\|_{L^\infty(t, T; L^p(\Gamma))}^p + \|\vartheta\|_{L^{p-2}(t, T; L^{2p-4}(\Gamma))}^{p-2} + \|\vartheta\|_{L^{p-2}(t, T; L^{3p-6}(\Omega))}^{p-2} \\ & \leq \|\vartheta(t)\|_p^p + \|\vartheta(t)\|_{p, \Gamma}^p + cpF\|\vartheta\|_{L^{2(p-1)}(t, T; L^{r(p-1)}(\Omega))}^{p-1} + c\|\vartheta\|_{L^{p-2}(t, T; L^{\frac{p-2}{2}}(\Omega))}^{p-2} \\ & \leq \|\vartheta(t)\|_p^p + \|\vartheta(t)\|_{p, \Gamma}^p + c(pF+1)\|\vartheta\|_{L^{2(p-1)}(t, T; L^{r(p-1)}(\Omega))}^{p-1}. \end{aligned} \quad (3.52)$$

Now, we perform Moser iterations as in the proof of Lemma 3.1, starting from $p_0 = 4 + \epsilon$. Then, in place of (3.18), we get the system

$$\begin{cases} \frac{1-\rho}{p_i-2} = \frac{1}{2(p_{i+1}-1)}, \\ \frac{\rho}{p_i} + \frac{1-\rho}{3(p_i-2)} = \frac{1}{r(p_{i+1}-1)}, \end{cases} \quad (3.53)$$

whence one computes, similarly as before,

$$K_{\epsilon'} = \frac{9 + 5\epsilon'}{9 + 3\epsilon'}. \quad (3.54)$$

and, finally,

$$\begin{aligned} p_{i+1} &= \frac{1}{2} \frac{p_i - 2}{1 - \rho} + 1 = \frac{1}{2} \left(1 + \frac{p_i}{p_i - 2} K_{\epsilon'} \right) (p_i - 2) + 1 \\ &= \frac{K_{\epsilon'} + 1}{2} p_i = \frac{9 + 4\epsilon'}{9 + 3\epsilon'} p_i =: H p_i = H^{i+1} p_0. \end{aligned} \quad (3.55)$$

Note that, as before, H is larger than 1 since $\epsilon' > 0$.

Then, we can follow with minor variations the proof of Lemma 3.1 up to formula (3.27). In particular, J_i is now defined by setting

$$J_i^{p_i} := \|\vartheta\|_{L^\infty(\tau_i, T; L^{p_i}(\Omega))}^{p_i} + \|\vartheta\|_{L^\infty(\tau_i, T; L^{p_i}(\Gamma))}^{p_i} + \|\vartheta\|_{L^{p-2}(\tau_i, T; L^{2p_i-4}(\Gamma))}^{p_i} + \|\vartheta\|_{L^{p-2}(\tau_i, T; L^{3p_i-6}(\Omega))}^{p_i}. \quad (3.56)$$

Then, the analogue of (3.28) reads

$$\begin{aligned} & \|\vartheta(\tau_{i+1})\|_{p_{i+1}}^{p_i} + \|\vartheta(\tau_{i+1})\|_{p_{i+1}, \Gamma}^{p_i} \leq \frac{1}{t_{i+1}} \int_{\tau_i}^{\tau_i + t_{i+1}} \left(\|\vartheta(t)\|_{p_{i+1}}^{p_i} + \|\vartheta(t)\|_{p_{i+1}, \Gamma}^{p_i} \right) dt \\ & \leq c \frac{i^2}{\tau} \int_{\tau_i}^{\tau_i + t_{i+1}} \left(\|\vartheta(t)\|_{3(p_i-2)}^{p_i} + \|\vartheta(t)\|_{2(p_i-2), \Gamma}^{p_i} \right) dt \leq c \frac{i^2}{\tau} J_i^{p_i}, \end{aligned} \quad (3.57)$$

which holds provided that $p_{i+1} \leq 2(p_i - 2)$. As p_{i+1} is given by (3.55), this means that we need

$$(4 + \epsilon)H^{i+1} \leq 2((4 + \epsilon)H^i - 2). \quad (3.58)$$

which is easily shown to be true for every $i \geq 0$ provided that we choose $\epsilon' \in (0, \epsilon]$ so small that

$$(4 + \epsilon)H = (4 + \epsilon) \frac{9 + 4\epsilon'}{9 + 3\epsilon'} \leq 2((4 + \epsilon) - 2). \quad (3.59)$$

Actually, also the choice $\epsilon' = \epsilon$ works. Notice that, since we used $p_i < p_{i+1} \leq 2(p_i - 2)$, it follows that the regularity (3.48) is needed in order for the iteration scheme to work. From this point on, the proof proceeds once again similarly with that of Lemma 3.1, hence we can omit the details. \blacksquare

As before, we have a better result in case the “initial” value of ϑ is uniformly bounded. The proof follows the preceding ones up to straightforward modifications.

Corollary 3.6. *Let $S \in \mathbb{R}$, $T > S$, $\alpha \geq 0$, $\beta \geq 0$. Let (u, ϑ) be a couple of sufficiently smooth functions solving in a suitable sense system (3.2)-(3.3) over (S, T) . Let also (3.43)-(3.44) hold, together with*

$$\vartheta(S) \in L^\infty(\Omega), \quad \alpha \vartheta(S) \in L^\infty(\Gamma), \quad \|\vartheta(S)\|_{L^\infty(\Omega)} + \alpha \|\vartheta(S)\|_{L^\infty(\Gamma)} =: \Theta. \quad (3.60)$$

Then, we have that

$$\|\vartheta\|_{L^\infty((S, T) \times \Omega)} \leq Q(L, F, \Theta). \quad (3.61)$$

3.3 Further regularity of time derivatives

We now prove that, for smooth solutions of Problem (P), boundedness of ϑ and u imply some L^p -regularity of time derivatives at least for strictly positive times. The proof of this result is a little bit tricky in the case $\beta > 0$ since the diffusion operators on Ω and on Γ act on different functions. In particular, we cannot treat the case when $\beta > 0$ and $\alpha = 0$ because in the derivation of the estimates the contribution of the boundary term ϑ_t is explicitly needed.

Lemma 3.7. *Let $S \in \mathbb{R}$, $T > S$, $\alpha \geq 0$, $\beta \geq 0$ and let (3.1) hold for some $\epsilon \in (0, 1)$. If $\beta > 0$, then let $\alpha > 0$. Let (u, ϑ) be a couple of sufficiently smooth functions solving in a suitable sense system (3.2)-(3.3) over the time interval (S, T) . Moreover, let*

$$u \in L^2(S, T; V), \quad \beta \vartheta \in L^2(S, T; V_\Gamma), \quad \|u\|_{L^2(S, T; V)} + \beta \|\vartheta\|_{L^2(S, T; V_\Gamma)} \leq U, \quad (3.62)$$

$$f \in L^2(S, T; H), \quad \|f\|_{L^2(S, T; H)} =: F, \quad (3.63)$$

$$u, \vartheta \in L^\infty((S, T) \times \Omega), \quad \|u\|_{L^\infty((S, T) \times \Omega)} + \|\vartheta\|_{L^\infty((S, T) \times \Omega)} \leq U, \quad (3.64)$$

for some (given) constants $F > 0$, $U > 0$. Then, for any $\tau \in (0, \min\{1, T - S\})$, we have that

$$\|\vartheta_t\|_{L^2(S+\tau, T; H)} + \alpha \|\vartheta_t\|_{L^2(S+\tau, T; H_\Gamma)} + \|u\|_{L^\infty(S+\tau, T; V)} + \beta \|\vartheta\|_{L^\infty(S+\tau, T; V_\Gamma)} \leq Q(F, U, \tau^{-1}). \quad (3.65)$$

PROOF. We just consider the case when $\alpha, \beta > 0$, which is the most difficult one. For simplicity, we can then set $\alpha = \beta = 1$ and write

$$\int_{\overline{\Omega}} v := \int_{\Omega} v + \int_{\Gamma} v, \quad (3.66)$$

whenever v is, say, an element of \mathcal{H} . Then, we test both (3.2) and (3.3) by $u_t = \vartheta_t / \vartheta^2$ (note that we are always assuming the solution to be smooth enough for our purposes). Then, standard integrations by parts lead to

$$\frac{1}{2} \frac{d}{dt} \left(\|\nabla u\|^2 + \int_{\Gamma} \frac{|\nabla_{\Gamma} \vartheta|^2}{\vartheta^2} \right) + \int_{\overline{\Omega}} \frac{\vartheta_t^2}{\vartheta^2} \leq \int_{\Gamma} \frac{\vartheta_t |\nabla_{\Gamma} \vartheta|^2}{\vartheta^3} + \int_{\Omega} f \frac{\vartheta_t}{\vartheta^2}. \quad (3.67)$$

Of course, using (3.64), the latter term can be simply estimated as follows:

$$\int_{\Omega} f \frac{\vartheta_t}{\vartheta^2} \leq \frac{1}{4} \int_{\Omega} \frac{\vartheta_t^2}{\vartheta^2} + Q(U) \|f\|^2. \quad (3.68)$$

The first term on the right hand side of (3.67) is more delicate. Actually, owing to standard interpolation inequalities, we have

$$\begin{aligned} \int_{\Gamma} \frac{\vartheta_t |\nabla_{\Gamma} \vartheta|^2}{\vartheta^3} &\leq Q(U) \|\vartheta_t\|_{\Gamma} \|\nabla_{\Gamma} \vartheta\|_{4, \Gamma}^2 \leq Q(U) \|\vartheta_t\|_{\Gamma} \|\vartheta\|_{H^2(\Gamma)} \|\vartheta\|_{\infty, \Gamma} \\ &\leq Q(U) (\|\vartheta_t\|_{\Gamma}^2 + \|\Delta_{\Gamma} \vartheta\|_{\Gamma}^2 + \|\vartheta\|_{\Gamma}^2) \\ &\leq Q(U) (\|\vartheta_t - \Delta_{\Gamma} \vartheta\|_{\Gamma}^2) + Q(U) - Q(U) \frac{d}{dt} \|\nabla_{\Gamma} \vartheta\|_{\Gamma}^2. \end{aligned} \quad (3.69)$$

Then, the last term is moved to the left hand side and will give a positive contribution. We now estimate the first term on the right hand side, which requires some work. Actually, we first notice that, comparing terms in equation (3.3), using the trace theorem and elliptic regularity results, we have, for arbitrarily small $\delta \in (0, 1/2)$,

$$\begin{aligned} Q(U) (\|\vartheta_t - \Delta_{\Gamma} \vartheta\|_{\Gamma}^2) + Q(U) \|\partial_n u\|_{\Gamma}^2 &\leq Q(U) \|\partial_n u\|_{\Gamma}^2 \leq Q(U) \|u\|_{H^{\frac{3}{2}+\delta}(\Omega)}^2 \\ &\leq Q(U) \left(\|u\|^2 + \|\Delta u\|_{H^{-\frac{1}{2}+\delta}(\Omega)}^2 + \|u\|_{H^{1+\delta}(\Gamma)}^2 \right) \\ &\leq Q(U) \left(1 + \|\Delta u\|^p \|u\|^{2-p} + \|u\|_{H^1(\Gamma)}^{2-2\delta} \|u\|_{H^2(\Gamma)}^{2\delta} \right), \end{aligned} \quad (3.70)$$

for a suitable $p \in (0, 2)$ depending on the choice of δ . Then, using (3.2) and Young's inequality, we obtain

$$\begin{aligned} Q(U) \|\Delta u\|^p \|u\|^{2-p} &\leq Q(U) \|\Delta u\|^p \leq Q(U) \|\vartheta_t - f\|^p \\ &\leq \sigma \int_{\Omega} \frac{\vartheta_t^2}{\vartheta^2} + Q_{\sigma}(U) (1 + \|f\|^2). \end{aligned} \quad (3.71)$$

Moreover, comparing once more terms in (3.3) and noting in particular that

$$\Delta_{\Gamma} u = \operatorname{div}_{\Gamma} (u^2 \nabla_{\Gamma} \vartheta), \quad (3.72)$$

we arrive at

$$\begin{aligned} \|\Delta_{\Gamma} u\|_{\Gamma} &\leq Q(U) \|\Delta_{\Gamma} \vartheta\|_{\Gamma} + Q(U) \|\nabla_{\Gamma} \vartheta\|_{4,\Gamma}^2 \\ &\leq Q(U) \|\Delta_{\Gamma} \vartheta\| + Q(U) + Q(U) \|\Delta_{\Gamma} \vartheta\|_{\Gamma} \|\vartheta\|_{\infty,\Gamma} \\ &\leq Q(U) \|\Delta_{\Gamma} \vartheta\|_{\Gamma} + Q(U). \end{aligned} \quad (3.73)$$

Hence, the last term in (3.70) can be controlled this way:

$$\begin{aligned} Q(U) \|u\|_{H^1(\Gamma)}^{2-2\delta} \|u\|_{H^2(\Gamma)}^{2\delta} &\leq Q_{\sigma}(U) \|u\|_{H^1(\Gamma)}^2 + \sigma \|\Delta_{\Gamma} u\|_{\Gamma}^2 \\ &\leq Q_{\sigma}(U) \|u\|_{H^1(\Gamma)}^2 + \sigma Q(U) \|\Delta_{\Gamma} \vartheta\|_{\Gamma}^2 + \sigma Q(U) \\ &\leq Q_{\sigma}(U) \|u\|_{H^1(\Gamma)}^2 + \sigma Q(U) \|\vartheta_t\|_{\Gamma}^2 + \sigma Q(U) \|\partial_{\mathbf{n}} u\|_{\Gamma}^2 + \sigma Q(U) \\ &\leq Q_{\sigma}(U) \|\vartheta\|_{H^1(\Gamma)}^2 + \sigma Q(U) \int_{\Gamma} \frac{\vartheta_t^2}{\vartheta^2} + \sigma Q(U) \|\partial_{\mathbf{n}} u\|_{\Gamma}^2 + \sigma Q(U), \end{aligned} \quad (3.74)$$

where $\sigma > 0$ is small and Q_{σ} also depends on σ . In particular, taking σ small enough both in (3.71) and in (3.74), and collecting (3.68)-(3.74), (3.67) gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|\nabla u\|^2 + \int_{\Gamma} \frac{|\nabla_{\Gamma} \vartheta|^2}{\vartheta^2} + Q(U) \|\nabla_{\Gamma} \vartheta\|_{\Gamma}^2 \right) &+ \frac{1}{2} \int_{\Omega} \frac{\vartheta_t^2}{\vartheta^2} \\ &\leq Q(U) (\|f\|^2 + \|\vartheta\|_{H^1(\Gamma)}^2 + 1). \end{aligned} \quad (3.75)$$

Now, thanks to (3.62), for any τ as in the statement, we can choose $\mathfrak{t} \in (S, S + \tau)$ such that

$$\|\nabla u(\mathfrak{t})\|^2 + \|\nabla_{\Gamma} \vartheta(\mathfrak{t})\|_{\Gamma}^2 \leq Q(U, \tau^{-1}). \quad (3.76)$$

The thesis then follows by integrating (3.75) over (\mathfrak{t}, T) . ■

As before, for regular “initial” data, the property holds starting from the initial time:

Corollary 3.8. *Let $S \in \mathbb{R}$, $T > S$, $\alpha \geq 0$, $\beta \geq 0$ and let (3.1) hold for some $\epsilon \in (0, 1)$. If $\beta > 0$, then let $\alpha > 0$. Let (u, ϑ) be a couple of sufficiently smooth functions solving in a suitable sense system (3.2)-(3.3) over the time interval (S, T) . Moreover, let (3.62)-(3.64) hold together with*

$$u(S) \in V, \quad \beta \vartheta(S) \in V_{\Gamma}, \quad \|u(S)\|_V + \beta \|\vartheta(S)\|_{V_{\Gamma}} =: Z, \quad (3.77)$$

for some $Z > 0$. Then,

$$\|\vartheta_t\|_{L^2(S,T;H)} + \alpha \|\vartheta_t\|_{L^2(S,T;H_{\Gamma})} + \|u\|_{L^{\infty}(S,T;V)} + \beta \|\vartheta\|_{L^{\infty}(S,T;V_{\Gamma})} \leq Q(F, U, Z). \quad (3.78)$$

4 Existence for smooth data

Based on the previous lemmas, we will prove here that, for sufficiently regular initial data and source term, Problem (P) has a global solution in a rather good regularity class.

4.1 Local existence of smooth solutions

We start by showing that, for regular data, a local in time smooth solution exists. For the sake of simplicity, we give the proof only in the case when both α and β are strictly positive. Actually, the other cases can be treated with differences that are mainly of technical character.

Theorem 4.1. *Let $\alpha > 0$, $\beta > 0$. Let $T > 0$ and let*

$$f \in C^0([0, T] \times \overline{\Omega}), \quad (4.1)$$

$$\vartheta_0 \in H^2(\Omega), \quad \underline{\vartheta} \leq \vartheta_0(x) \leq \overline{\vartheta} \text{ for all } x \in \overline{\Omega}, \quad (4.2)$$

for suitable constants $0 < \underline{\vartheta} < \overline{\vartheta}$. Then, there exists $T_0 \in (0, T]$ depending on f and ϑ_0 (and in particular on $\underline{\vartheta}, \overline{\vartheta}$), such that Problem (P) admits a solution (ϑ, u) over the time interval $[0, T_0]$ satisfying

$$\vartheta \in H^1(0, T_0; \mathcal{H}) \cap C^0([0, T_0] \times \overline{\Omega}), \quad \frac{\underline{\vartheta}}{2} \leq \vartheta(t, x) \leq 2\overline{\vartheta} \text{ for all } (t, x) \in [0, T_0] \times \overline{\Omega}, \quad (4.3)$$

$$u \in L^2(0, T_0; H^2(\Omega)), \quad \vartheta \in L^2(0, T_0; H^2(\Gamma)). \quad (4.4)$$

PROOF. We only sketch it, since it follows from more or less classical arguments for quasilinear parabolic problems. The key idea is to regularize the monotone function $\gamma(r) = -1/r$ by introducing a function $\gamma_R : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties

$$\gamma_R \in C^2(\mathbb{R}), \quad \gamma_R'' \in L^\infty(\mathbb{R}), \quad \gamma_R(r) = \gamma(r) \quad \forall r \in \left[\frac{\underline{\vartheta}}{2}, 2\overline{\vartheta}\right], \quad (4.5)$$

$$\exists 0 < m_R < M_R \text{ such that } m_R \leq \gamma_R'(r) \leq M_R \quad \forall r \in \mathbb{R}. \quad (4.6)$$

In other words, γ_R coincides with γ over an interval that is strictly larger than the range of the initial values of the problem. Outside that interval, we substitute it by a smooth and bi-Lipschitz approximation (e.g., we can take the first order Taylor expansion and then mollify). At this point, we can solve the problem by means of a fixed point argument. We just give the highlight of this procedure and set, for simplicity, $\alpha = \beta = 1$.

As a first step, we take a function ϕ in the ball Φ of radius 1 of the space $L^2(0, T_0; H^\varepsilon(\Gamma))$, where $\varepsilon \in (0, 1/2)$ is fixed, but arbitrary. Then, we solve the initial-value problem

$$\eta_t - \Delta_\Gamma \eta = \phi, \quad \eta|_{t=0} = \vartheta_0|_\Gamma, \quad \text{on } \Gamma. \quad (4.7)$$

Thanks to classical results on parabolic equations, the solution η is unique and lies in a suitable ball of radius $Q_1(\|\vartheta_0\|_{H^2(\Omega)})$ of the space

$$H^1(0, T_0; H^\varepsilon(\Gamma)) \cap L^2(0, T_0; H^{2+\varepsilon}(\Gamma)). \quad (4.8)$$

Note that, indeed, the boundary initial datum lies in $H^{3/2}(\Gamma)$ by (4.2) and the trace theorem. We also remark that, by interpolation, the space (4.8) is continuously embedded into $C^{0,a}([0, T_0] \times \Gamma)$ for some $a > 0$ depending on the choice of ε (recall that Ω is smooth and its boundary Γ is a 2-dimensional manifold). Thus, we can take T_0 so small that, additionally,

$$\frac{3\underline{\vartheta}}{4} \leq \eta(t, x) \leq \frac{3\overline{\vartheta}}{2} \text{ for all } (t, x) \in [0, T_0] \times \Gamma. \quad (4.9)$$

As a subsequent step, we solve the problem

$$\vartheta_t - \Delta \gamma_R(\vartheta) = f, \quad \vartheta|_{t=0} = \vartheta_0, \quad \text{in } \Omega, \quad (4.10)$$

$$\gamma_R(\vartheta) = \gamma_R(\eta), \quad \text{on } \Gamma. \quad (4.11)$$

This is a quasilinear parabolic system with bi-Lipschitz continuous nonlinearity γ_R and Dirichlet boundary conditions. Standard methods permit then to verify that it admits a unique solution ϑ such that

$$\vartheta \in H^1(0, T_0; H), \quad \gamma_R(\vartheta) \in L^2(0, T_0; H^2(\Omega)). \quad (4.12)$$

For instance, the basic a priori estimate corresponding to the above regularity can be obtained testing the equation in (4.10) by the function $\partial_t(\gamma_R(\vartheta) - \mathcal{R}(\gamma_R(\eta)))$, where \mathcal{R} denotes the harmonic extension operator, namely

$$-\Delta \mathcal{R}(v) = 0 \quad \text{in } \Omega, \quad \mathcal{R}(v) = v \quad \text{on } \Gamma, \quad (4.13)$$

where v is, say, a function in H_Γ . Note that the regularity (4.8) of the trace and the C^2 -regularity of γ_R are essential for the sake of obtaining (4.12), as a direct check permits to verify.

Then, being γ_R bi-Lipschitz, $\gamma_R(\vartheta)$ lies in a bounded ball of radius $Q_2(\|\vartheta_0\|_{H^2(\Omega)})$ of the space

$$H^1(0, T_0; H) \cap L^2(0, T_0; H^2(\Omega)). \quad (4.14)$$

Moreover, the equation in (4.10) is quasilinear and uniformly parabolic and it has uniformly bounded initial data (by (4.2)), boundary data (by (4.9)), and forcing term (by (4.1)). Thus, standard barrier arguments entail that, up to possibly taking a smaller initial time T_0 (in a way that only depends on the known norms of ϑ_0 and f , on the regularized function γ_R , and on the truncation values $\underline{\vartheta}, \overline{\vartheta}$),

$$\frac{\vartheta}{2} \leq \vartheta(t, x) \leq 2\overline{\vartheta} \quad \text{for all } (t, x) \in [0, T_0] \times \overline{\Omega}. \quad (4.15)$$

Consequently, by (4.14), interpolation, and the trace theorem, the trace function $\partial_{\mathbf{n}}\gamma(\vartheta)$ lies in a bounded ball of radius $Q_3(\|\vartheta_0\|_{H^2(\Omega)})$ of the space

$$L^p(0, T_0; H^\varepsilon(\Gamma)) \quad \text{for some } p > 2. \quad (4.16)$$

In particular, it is possible to take T_0 so small that the map

$$\mathcal{T} : \Phi \rightarrow L^2(0, T_0; H^\varepsilon(\Gamma)), \quad \mathcal{T} : \phi \mapsto \partial_{\mathbf{n}}\gamma(\vartheta), \quad (4.17)$$

takes values into Φ . Moreover, thanks to (4.14) and standard embedding and trace theorems, the map \mathcal{T} is compact. Finally, a number of standard checks permit to see that it is continuous. Hence, the Schauder fixed point theorem applies to \mathcal{T} . This gives a local solution to the system

$$\vartheta_t - \Delta \gamma_R(\vartheta) = f, \quad \text{in } \Omega, \quad (4.18)$$

$$\eta_t - \Delta_\Gamma \eta = -\partial_{\mathbf{n}}\gamma_R(\vartheta), \quad \text{on } \Gamma, \quad (4.19)$$

plus the initial conditions and the boundary condition $\vartheta|_\Gamma = \eta$. However, thanks to (4.15) and the latter (4.5), $\gamma_R(\vartheta)$ coincides with $\gamma(\vartheta) = -1/\vartheta =: u$ everywhere in $[0, T_0] \times \overline{\Omega}$. Hence, (ϑ, u) is a local smooth solution to (3.2)-(3.3). This concludes the proof. \blacksquare

4.2 Energy estimate

In this section we prove the basic energy estimate satisfied by solutions of Problem (P). We start by recalling a generalized version of Poincaré's inequality (see for instance [15, Lemma 3.2] for a proof).

Lemma 4.2. *Assume Ω is a bounded open subset of \mathbb{R}^d . Suppose $v \in W^{1,1}(\Omega)$ and $v \geq 0$ a.e. in Ω . Then, setting $K := \int_\Omega (\log v)^+$, the following estimate holds:*

$$\|v\|_{L^1(\Omega)} \leq |\Omega|e^{C_1 K} + \frac{C_2}{|\Omega|} \|\nabla v\|_{L^1(\Omega)}, \quad (4.20)$$

the positive constants C_1 and C_2 depending only on Ω .

Lemma 4.3 (Energy estimate). *Let $\alpha \geq 0$, $\beta \geq 0$ and let (2.10)-(2.11) and (2.13) hold. Let (ϑ, u) be a sufficiently smooth solution to Problem (P). Then, we have that*

$$\mathcal{E}(t) + \int_0^t (\|\nabla u\|^2 + \beta \|\nabla_\Gamma \log \vartheta\|_\Gamma^2) + \|u\|_{L^2(t, t+1; V)}^2 \leq Q(\mathbb{E}_0), \quad \forall t \geq 0. \quad (4.21)$$

PROOF. Test (2.6) and (2.7) with $1 + u$. This gives

$$\frac{d}{dt}\mathcal{E}(t) + (\|\nabla u\|^2 + \beta\|\nabla_{\Gamma} \log \vartheta\|_{\Gamma}^2) = \int_{\Omega} f(1 + u). \quad (4.22)$$

Then, using (2.13) and the Poincaré-Wirtinger inequality, we estimate the right hand side as follows:

$$\int_{\Omega} f(1 + u) = \int_{\Omega} fu = \int_{\Omega} f(u - m_{\Omega}(u)) \leq \frac{1}{2}\|\nabla u\|^2 + c\|f\|_{L^{6/5}(\Omega)}^2. \quad (4.23)$$

Integrating (4.22) in time from 0 to t , we obtain

$$\mathcal{E}(t) + \int_0^t \left(\frac{1}{2}\|\nabla u\|^2 + \beta\|\nabla_{\Gamma} \log \vartheta\|_{\Gamma}^2 \right) \leq \mathbb{E}_0 + c\|f\|_{L^2(0,T;L^{6/5}(\Omega))}^2, \quad \forall t \geq 0. \quad (4.24)$$

To obtain the control on the full V -norm of u , we use Lemma 4.2. Thus, taking $v = 1/\vartheta = -u$ in (4.20), we infer

$$\|u\|_{L^1(\Omega)} \leq |\Omega|e^{C_1 \int_{\Omega} \log^{-} \vartheta} + \frac{C_2}{|\Omega|}\|\nabla u\|_{L^1(\Omega)}. \quad (4.25)$$

Hence, squaring, integrating over $(t, t+1)$, and using (4.24), we obtain

$$\|u\|_{L^2(t,t+1;L^1(\Omega))} \leq Q(\mathbb{E}_0). \quad (4.26)$$

The thesis follows combining (4.26) and (4.24). \blacksquare

4.3 Proof of global existence

Our next aim is to prove that, for smooth initial data, the solution constructed in Section 4.1 has, in fact, a global in time character. We can treat all cases with the exception of $\alpha = 0, \beta > 0$. This case will be dealt with separately in Section 5.3 below.

Theorem 4.4. *Let $\alpha \geq 0, \beta \geq 0$ and, if $\alpha = 0$, then let $\beta = 0$. Let (2.13) hold together with*

$$\vartheta_0 \in \mathcal{V}, \quad \underline{\vartheta} \leq \vartheta_0(x) \leq \overline{\vartheta} \text{ for all } x \in \overline{\Omega}, \quad (4.27)$$

for some $0 < \underline{\vartheta} < \overline{\vartheta}$. Then, there exists a global solution to Problem (P) satisfying, for all $T > 0$,

$$\vartheta \in H^1(0, T; H) \cap L^\infty((0, T) \times \Omega) \cap L^\infty(0, T; V), \quad (4.28)$$

$$u \in H^1(0, T; H) \cap L^\infty((0, T) \times \Omega) \cap L^\infty(0, T; V) \cap L^2(0, T; H^{3/2}(\Omega)), \quad (4.29)$$

$$\alpha\eta \in H^1(0, T; H_{\Gamma}), \quad \beta\eta \in L^\infty(0, T; V_{\Gamma}) \cap L^2(0, T; H^2(\Gamma)). \quad (4.30)$$

Moreover, if either $\alpha > 0$ and $\beta > 0$ or $\alpha = \beta = 0$, then we have more precisely

$$\vartheta \in L^2(0, T; H^2(\Omega)), \quad u \in L^2(0, T; H^2(\Omega)). \quad (4.31)$$

PROOF. Let us fix (an arbitrary) $T > 0$. In order to get a local solution via Theorem 4.1 we need to construct sequences of regularized initial and source data $\{\vartheta_{0,n}\}$ and $\{f_n\}$ satisfying, for all $n \in \mathbb{N}$, (4.1) and, respectively, (4.2). Moreover, we need $\vartheta_{0,n} \rightarrow \vartheta_0$ and $f_n \rightarrow f$ in proper ways. The details of the regularization procedure are sketched in Section 5.2 below, to which we refer the reader.

Then, thanks to Theorem 4.1, for all $n > 0$ we have a solution to the n -Problem defined on some interval $(0, T_{0,n})$, with $T_{0,n} \leq T$. We now deduce global in time estimates independent of n and, for the sake of simplicity, we shall directly work on the time interval $(0, T)$ rather than on $(0, T_{0,n})$. As usual, this can be justified *a posteriori* by means of standard extension arguments.

First of all, we can apply Lemma 4.3, which gives the estimates

$$\|\vartheta_n\|_{L^\infty(0,T;L^1(\Omega))} + \alpha\|\eta_n\|_{L^\infty(0,T;L^1(\Gamma))} + \|u_n\|_{L^2(0,T;V)} + \beta\|\log \eta_n\|_{L^2(0,T;V_{\Gamma})} \leq c, \quad (4.32)$$

for $c > 0$ independent of n . Indeed, from this point on, we go back to the notation η_n when we indicate the boundary value of ϑ_n . Next, we can apply Corollaries 3.2, 3.6 (with $S = 0$), which give

$$\|u_n\|_{L^\infty((0,T)\times\Omega)} + \|\vartheta_n\|_{L^\infty((0,T)\times\Omega)} \leq c. \quad (4.33)$$

Clearly, the same uniform boundedness properties hold also for the traces on Γ .

Next, by Corollary 3.8 (here the restriction on α and β comes into play), we have

$$\|\vartheta_n\|_{H^1(0,T;H)} + \alpha\|\eta_n\|_{H^1(0,T;H_\Gamma)} + \|u_n\|_{L^\infty(0,T;V)} + \beta\|\eta_n\|_{L^\infty(0,T;V_\Gamma)} \leq c. \quad (4.34)$$

Comparing terms in equation (3.2), we also obtain

$$\|\Delta u_n\|_{L^2(0,T;H)} \leq c. \quad (4.35)$$

Moreover, in the case when $\beta > 0$ (and hence $\alpha > 0$), we observe that

$$u_n|_\Gamma = -\frac{1}{\eta_n} \quad \text{and} \quad \left\| \frac{1}{\eta_n} \right\|_{L^2(0,T;V_\Gamma)} \leq c, \quad (4.36)$$

the latter bound following from (4.32)-(4.33). Hence, by standard regularity results for elliptic problems with Dirichlet boundary conditions (see, e.g., [6, Theorem 3.1.5]), we obtain

$$\|u_n\|_{L^2(0,T;H^{3/2}(\Omega))} \leq c. \quad (4.37)$$

Thanks to (4.32) and (4.35), we can apply the trace theorem [6, Theorem 2.7.7], which yields

$$\|\partial_n u_n\|_{L^2(0,T;H_\Gamma)} \leq c. \quad (4.38)$$

Consequently, using the regularity of the boundary initial datum (4.27) we obtain

$$\|\eta_{n,t}\|_{L^2(0,T;H_\Gamma)} + \|\eta_n\|_{L^2(0,T;H^2(\Gamma))} \leq c. \quad (4.39)$$

Next, we observe that

$$\Delta_\Gamma \left(-\frac{1}{\eta_n} \right) = \frac{1}{\eta_n^2} \Delta_\Gamma \eta_n - 2 \frac{|\nabla_\Gamma \eta_n|^2}{\eta_n^3}. \quad (4.40)$$

Thus, using the boundary analogue of (4.33), (4.39), and the Gagliardo-Nirenberg inequality

$$\|v\|_{W^{1,4}} \leq c\|v\|_{H^2}^{1/2} \|v\|_{L^\infty}^{1/2} + c\|v\|_{L^\infty}^{1/2}, \quad (4.41)$$

which gives that $|\nabla_\Gamma \eta_n|^2 \in L^2(\Gamma)$ uniformly w.r.t. n , we readily obtain that

$$\left\| \frac{1}{\eta_n} \right\|_{L^2(0,T;H^2(\Gamma))} \leq c, \quad (4.42)$$

whence (4.37) can be improved to

$$\|u_n\|_{L^2(0,T;H^2(\Omega))} \leq c \quad (4.43)$$

and the same bound holds for $\{\vartheta_n\}$ thanks again to (4.33) and (4.41).

Let us now deal with the case when $\beta = 0$. Then, if it is also $\alpha = 0$, we are just dealing with no-flux conditions. Hence, from (4.35) we directly deduce (4.43) and, consequently, (4.31). If, instead, $\beta = 0$ and $\alpha > 0$, we then have that $\partial_n u_n = -\alpha \eta_{n,t}$, so that, from (4.34), (4.35), and regularity results for Neumann elliptic boundary value problems, we deduce once more (4.37). However, it does not seem possible to arrive at (4.43) since we do not have sufficient regularity of $\partial_n u_n$.

In all cases, standard applications of the Aubin-Lions lemma permit to take the limit $n \nearrow \infty$ and obtain (up to the extraction of a subsequence) existence of a triplet (ϑ, η, u) solving Problem (P) and complying with the regularity properties (4.28)-(4.30) and, possibly, (4.31). In particular, it is worth noting that, in the case $\alpha = \beta = 0$ there is no boundary function η . Otherwise, η coincides with the trace of ϑ thanks to the quoted regularity properties and to continuity of the trace operator. The proof is concluded. \blacksquare

5 Weak solutions

5.1 Existence for finite-energy data

Theorem 5.1. *Let assumptions (2.10)-(2.11) and (2.13) hold and let us assume that, if $\alpha = 0$, then $\beta = 0$. Then, Problem (P) admits at least one energy solution (ϑ, u) satisfying the uniform energy estimate*

$$\mathcal{E}(t) + \int_0^t \left(\frac{1}{2} \|\nabla u\|^2 + \beta \|\nabla_\Gamma \log \vartheta\|_\Gamma^2 \right) \leq \mathbb{E}_0 + c \|f\|_{L^2(0,t;L^{6/5}(\Omega))}^2, \quad \forall t \geq 0. \quad (5.1)$$

and the further regularity property

$$\|u(t)\|_{L^\infty(\Omega)} \leq Q(\mathbb{E}_0, M_\epsilon, \tau^{-1}) \quad \text{for a.e. } t \geq \tau, \quad \tau \in (0, 1), \quad (5.2)$$

where \mathbb{E}_0 represents the “initial energy”

$$\mathbb{E}_0 := \mathcal{E}(\vartheta_0, \eta_0) \quad (5.3)$$

and we have set

$$M_\epsilon := \|f\|_{L^2(0,+\infty;L^{6/5}(\Omega))} + \|f\|_{\mathcal{T}^2(0,+\infty;L^{3+\epsilon}(\Omega))}. \quad (5.4)$$

PROOF. As before, we start with the case when α and β are both strictly positive. The other cases can be treated with small variants that will be outlined at the end of the proof. In order to apply Theorem 4.4, we consider a sequence of initial data $\{\vartheta_{0,n}\}$ such that

$$\vartheta_{0,n} \in \mathcal{V}, \quad 0 < \vartheta_n \leq \vartheta_{0,n}(x) \leq \bar{\vartheta}_n < +\infty, \quad \text{a.e. in } \Omega \text{ and on } \Gamma, \quad (5.5)$$

$$\vartheta_{0,n} \rightarrow \vartheta_0 \quad \text{strongly in } L^1(\Omega), \quad \vartheta_{0,n}|_\Gamma \rightarrow \eta_0 \quad \text{strongly in } L^1(\Gamma), \quad (5.6)$$

as $n \nearrow \infty$. The construction of such a sequence is sketched in Section 5.2 below. Notice that we do not need to approximate f . Then, for any $n \geq 0$, the assumptions of Theorem 4.4 are fulfilled and we have existence of a “smooth” solution (ϑ_n, u_n) . Moreover, we can still use the energy estimate, which gives the analogue of (4.32). As a further consequence, we can apply Lemma 3.1 on the generic time interval $(t, t+1)$, so that, combining (4.21) with (3.6), we obtain

$$\|u_n(t)\|_{L^\infty(\Omega)} \leq Q(\mathbb{E}_0, \tau^{-1}) \quad \text{for a.e. } t \geq \tau > 0, \quad (5.7)$$

where it is worth noting once more that Q does not depend on the final time T . In other words, we have a uniform bound.

At this point, we use L^1 -techniques in order to take the limit $n \nearrow \infty$. To this aim, we work on the generic interval $(0, T)$ and rewrite the approximate equation (2.6) in the equivalent form (which is possible since u_n has the good regularity properties (4.29))

$$\partial_t u_n = u_n^2 \Delta u_n + u_n^2 f. \quad (5.8)$$

Then, we test (5.8) with $\varphi \in L^\infty(0, T; H_0^s(\Omega))$, where we choose $s > \frac{3}{2}$ so that $H_0^s(\Omega) \subset C^0(\bar{\Omega})$, continuously. Note that, since $\varphi = 0$ on Γ , we do not have to take into account the contribution of the boundary. Using (4.21) and (5.7), it is a standard matter to get

$$\|u_{n,t}\|_{L^1(\tau, T; H^{-s})} \leq c. \quad (5.9)$$

Next, for any $\varepsilon > 0$ and $r \in \mathbb{R}$ we introduce the approximate sign function as $\text{sign}_\varepsilon(v) := \frac{v}{\varepsilon + |v|}$. Then, we write the approximate equations (2.6) and (2.7) for the indexes m and n , take the differences and test them, respectively, by $\text{sign}_\varepsilon(u_m - u_n)$ and by $\text{sign}_\varepsilon(u_{m,\Gamma} - u_{n,\Gamma})$. Indeed, for any $m, n \in \mathbb{N}$ and $\varepsilon > 0$, we have

$$\text{sign}_\varepsilon(u_m - u_n) \in L^2(0, T; V_\Gamma) \quad (5.10)$$

since sign_ε is Lipschitz. Then, we arrive at

$$\begin{aligned} & ((\vartheta_{m,t} - \vartheta_{n,t}), \text{sign}_\varepsilon(u_m - u_n)) + \alpha((\eta_{m,t} - \eta_{n,t}), \text{sign}_\varepsilon(u_m - u_n))_\Gamma \\ & - \beta(\Delta_\Gamma(\eta_m - \eta_n), \text{sign}_\varepsilon(u_m - u_n))_\Gamma \leq 0. \end{aligned} \quad (5.11)$$

Here, we used integration by parts to prove that

$$-(\Delta(u_m - u_n), \text{sign}_\varepsilon(u_m - u_n)) \geq -(\partial_n(u_m - u_n), \text{sign}_\varepsilon(u_m - u_n))_\Gamma, \quad (5.12)$$

which is possible thanks to the good regularity of approximate solutions, to monotonicity of sign_ε , and to (5.10). Thus, integrating (5.10) over $(0, t)$, for a generic $t \in (0, T]$, we obtain

$$\begin{aligned} & \int_0^t ((\vartheta_{m,t} - \vartheta_{n,t}), \text{sign}_\varepsilon(u_m - u_n)) + \alpha \int_0^t ((\eta_{m,t} - \eta_{n,t}), \text{sign}_\varepsilon(u_m - u_n))_\Gamma \\ & - \beta \int_0^t (\Delta_\Gamma(\eta_m - \eta_n), \text{sign}_\varepsilon(u_m - u_n))_\Gamma \leq 0. \end{aligned} \quad (5.13)$$

Next, we take the limit $\varepsilon \searrow 0$ and notice that, since $v \mapsto -1/v$ is monotone increasing, it turns out that $\text{sign}_0(\vartheta_m - \vartheta_n) = \text{sign}_0(u_m - u_n)$ almost everywhere in $(0, T) \times \Omega$ and $\text{sign}_0(\eta_m - \eta_n) = \text{sign}_0(u_m - u_n)$ almost everywhere in $(0, T) \times \Gamma$, where sign_0 denotes the sign-like function with $\text{sign}_0(0) = 0$. Then,

$$\begin{aligned} & \int_0^t ((\vartheta_{m,t} - \vartheta_{n,t}), \text{sign}_0(\vartheta_m - \vartheta_n)) + \alpha \int_0^t ((\eta_{m,t} - \eta_{n,t}), \text{sign}_0(\eta_m - \eta_n))_\Gamma \\ & - \beta \int_0^t (\Delta_\Gamma(\eta_m - \eta_n), \text{sign}_0(\eta_m - \eta_n))_\Gamma \leq 0. \end{aligned} \quad (5.14)$$

Applying the Brezis-Strauss lemma (see [5, Lemma 2]), we can integrate by parts the boundary Laplacian, which gives a nonnegative contribution. Thus, we arrive at

$$\|\vartheta_m(t) - \vartheta_n(t)\|_1 + \alpha \|\eta_m(t) - \eta_n(t)\|_{1,\Gamma} \leq \|\vartheta_{0,m} - \vartheta_{0,n}\|_1 + \alpha \|\vartheta_{0,m} - \vartheta_{0,n}\|_{1,\Gamma}. \quad (5.15)$$

Taking the supremum for $t \in [0, T]$, we finally obtain

$$\vartheta_n \rightarrow \vartheta \quad \text{strongly in } C^0([0, T]; L^1(\Omega)), \quad \alpha \eta_n \rightarrow \alpha \eta \quad \text{strongly in } C^0([0, T]; L^1(\Gamma)), \quad (5.16)$$

for suitable limit functions ϑ and η . Moreover, using (4.21), (5.9), and the Aubin-Lions lemma, we have that, up to the extraction of a subsequence,

$$u_n \rightarrow u \quad \text{weakly in } L^2(0, T; V) \quad \text{and strongly in } L^2(\tau, T; H^{1-\delta}(\Omega)), \quad (5.17)$$

for all $\tau \in (0, 1)$, $\delta \in (0, 1)$, and for some limit function u . In particular, taking a test function ξ satisfying (2.18), we can test both (2.6) and (2.7) (at the n -level) by ξ and integrate by parts. This gives the n -analogue of (2.19). Moreover, the convergence properties proved above permit to take the limit $n \nearrow \infty$ to show that (2.19) holds also at the limit level. Hence, to conclude the proof it remains to identify the functions ϑ and η in terms of u . To do this, we first observe that, combining the first (5.16) with (5.17), it follows that $\vartheta = -1/u$ almost everywhere in $(0, T) \times \Omega$.

Next, by (5.17) and continuity of the trace operator, we have, for all $\tau \in (0, T)$,

$$(u_n)_\Gamma \rightarrow u_\Gamma \quad \text{strongly in } L^2(\tau, T; H^{1/2-\delta}(\Gamma)), \quad (5.18)$$

with u_Γ denoting the trace of u . In particular, thanks to arbitrariness of τ , pointwise convergence holds on $(0, T) \times \Gamma$, up to the extraction of a further subsequence. Being $\eta_n = -1/(u_n)_\Gamma$ almost everywhere in $(0, T) \times \Gamma$ and for all $n \in \mathbb{N}$, we then deduce that

$$\eta_n \rightarrow -\frac{1}{u_\Gamma} \quad \text{a.e. on } (0, T) \times \Gamma. \quad (5.19)$$

Combining this with the second (5.16), we then obtain relation (2.17). This concludes the proof of the Theorem in the case $\alpha > 0$, $\beta > 0$.

Let us now consider the case $\alpha > 0$ and $\beta = 0$ (the case $\alpha = \beta = 0$ is essentially already known and, in any case, it is simpler to treat). Since Lemma 3.1 does not depend on whether β is zero or not, estimate (5.7) (and consequently (5.9)) still holds when $\beta = 0$. Moreover, it is easy to check that the Cauchy argument (5.10)-(5.16) can be reproduced also when $\beta = 0$. The proof of the Theorem is concluded. \blacksquare

Remark 5.2. Note that, in the case when $\alpha > 0$ and $\beta > 0$, η_0 needs not be the trace of ϑ_0 . Thus, there is a boundary layer in the sense specified by (5.18)-(5.19). However, (2.17) still make sense thanks to instantaneous regularization properties.

Remark 5.3. The reason for not considering the case where $\alpha = 0$ and $\beta > 0$ can be already seen in the Definition 2.1 of a weak solution. Indeed, if $\alpha = 0$ and $\beta > 0$, then the last integral on the first row of (2.19) may make no sense since ϑ needs not have any summability property on the boundary. On the other hand, if $\alpha > 0$, then that integral is finite thanks to the latter (2.14). Actually, in the case when $\alpha = 0$ and $\beta > 0$, we can still arrive at formal estimates of the form (5.1) and (5.2). However, these estimate do not suffice to prove existence exactly for the reason outlined above.

5.2 Approximation of data

In this part, we sketch the approximation of data needed in the proofs of Theorems 4.4 and 5.1. Since the procedures are rather standard, we just give the highlights without entering too much into details.

Approximation of data for Theorem 4.4. We start with the initial data. Let us given $\vartheta_0 \in \mathcal{V}$ such that $\underline{\vartheta} \leq \vartheta_0(x) \leq \overline{\vartheta}$ for all $x \in \overline{\Omega}$ (as in the statement, see (4.27)), and let η_0 be its trace on Γ . Since we want to apply Theorem 4.1 (cf., in particular, (4.2)), for any $n \in \mathbb{N}$ we need to have an approximate datum $\vartheta_{0,n}$ (with trace $\eta_{0,n}$) such that $\vartheta_{0,n} \in H^2(\Omega)$ for all n and

1. $\vartheta_{0,n} \xrightarrow{n \nearrow +\infty} \vartheta_0$ in \mathcal{V} ,
2. $\underline{\vartheta} \leq \vartheta_{0,n}(x) \leq \overline{\vartheta}$ for all $x \in \overline{\Omega}$.

To construct $\vartheta_{0,n}$ we then consider the elliptic problem

$$\begin{cases} \vartheta_{0,n} - \frac{1}{n} \Delta \vartheta_{0,n} = \vartheta_0 & \text{in } \Omega, \\ -\frac{1}{n} \partial_n \vartheta_{0,n} = -\frac{1}{n} \Delta_\Gamma \eta_{0,n} + \eta_{0,n} - \eta_0 & \text{on } \Gamma, \quad \vartheta_{0,n}|_\Gamma = \eta_{0,n}. \end{cases} \quad (5.20)$$

Then, the standard elliptic theory gives that, for any fixed n , $\vartheta_{0,n}$ lies in $H^2(\Omega)$ and $\eta_{0,n}$ lies in $H^2(\Gamma)$.

To prove the convergence $\vartheta_{0,n} \rightarrow \vartheta_0$, we test (5.20) $|_\Omega$ by $(\vartheta_{0,n} - \Delta \vartheta_{0,n})$ and, correspondingly, (5.20) $|_\Gamma$ by $(1 + n^{-1})\eta_{0,n} - \eta_0$. Then, by integrations by parts and Young's inequality, it is not difficult to obtain

$$\|\vartheta_{0,n}\|_{\mathcal{V}}^2 + 2n\|\eta_{0,n} - \eta_0\|_\Gamma^2 \leq \|\vartheta_0\|_{\mathcal{V}}^2,$$

which clearly implies the desired convergence. On the other hand, to prove uniform boundedness of $\vartheta_{0,n}$, we use a maximum principle argument. We prove only the upper bound, the lower one being completely equivalent. For a generic $M > \overline{\vartheta}$, we test (5.20) $|_\Omega$ by $(\vartheta_{0,n} - M)^+$. Integrating by parts the Laplacian and using (5.20) $|_\Gamma$, we then obtain

$$\int_\Omega \vartheta_{0,n}(\vartheta_{0,n} - M)^+ + \int_\Gamma \eta_{0,n}(\eta_{0,n} - M)^+ \leq \int_\Omega \vartheta_0(\vartheta_{0,n} - M)^+ + \int_\Gamma \eta_0(\eta_{0,n} - M)^+.$$

Adding to both sides the quantity $-M \int_\Omega (\vartheta_{0,n} - M)^+ - M \int_\Gamma (\eta_{0,n} - M)^+$, we arrive at

$$\|(\vartheta_{0,n} - M)^+\|^2 + \|(\eta_{0,n} - M)^+\|_\Gamma^2 \leq \int_\Omega (\vartheta_0 - M)(\vartheta_{0,n} - M)^+ + \int_\Gamma (\eta_0 - M)(\eta_{0,n} - M)^+ \leq 0. \quad (5.21)$$

This clearly gives the desired inequality $\vartheta_{0,n}(x) \leq \overline{\vartheta}$ for all $x \in \overline{\Omega}$.

Concerning the source term, recalling that f satisfies (2.13), a combination of truncation and mollification techniques, together with a suitable correction of the spatial mean values permits to construct a family of functions $\{f_n\}$, $n \in \mathbb{N}$, such that

$$f_n \in C^0([0, T] \times \overline{\Omega}), \quad \|f_n\|_{L^\infty((0, T) \times \Omega)} \leq n, \quad (5.22)$$

$$f_n \rightarrow f \quad \text{strongly in } L^1((0, T) \times \Omega), \quad (5.23)$$

$$(f_n(t))_\Omega = 0 \quad \text{for all } t \in [0, T], \quad (5.24)$$

$$\|f_n\|_{L^2((0, T; L^{3+\epsilon}(\Omega)))} \leq c. \quad (5.25)$$

The details of this construction are left to the reader. Notice that (5.22)-(5.25) suffice both to apply Theorem 4.1 at the level n to get a local smooth solution and to perform the estimates of Theorem 4.4 uniformly in n in order to let $n \rightarrow \infty$.

Approximation of data for Theorem 5.1. We detail such an approximation just in the case $\alpha > 0$, which is a little bit trickier. So, let $(\vartheta_0, \eta_0) \in L^1(\Omega) \times L^1(\Gamma)$ such that $\log \vartheta_0 \in L^1(\Omega)$ and $\log \eta_0 \in L^1(\Gamma)$. We need to construct $\vartheta_{0,n}$ in such a way that properties (5.5)-(5.6) hold. Then, we consider first the function ϑ_0 . Using that Γ is smooth, we can first extend it (e.g., by reflection) to a neighbourhood Ω_* of $\bar{\Omega}$. It is then clear that the new function, note it as $\vartheta_{0,*}$, lies in $L^1(\Omega_*)$; moreover, $\log \vartheta_{0,*} \in L^1(\Omega_*)$. Next, we truncate $\vartheta_{0,*}$, setting

$$\vartheta_{0,n}^{(1)} := \min \left\{ \max \left\{ \vartheta_{0,*}, \frac{1}{n} \right\}, n \right\}, \quad \text{a.e. in } \Omega_*. \quad (5.26)$$

Finally, we regularize, setting $\vartheta_{0,n}^{(2)} := \rho_n * \vartheta_{0,n}^{(1)}$, where $\{\rho_n\}$ is a suitable sequence of smooth and compactly supported mollifiers. Then, straightforward checks (based on the properties of convolutions and on Lebesgue's theorem) and a standard diagonal argument permit to verify that $\vartheta_{0,n}^{(2)}$ is smooth and tends to ϑ_0 strongly in $L^1(\Omega)$. Moreover, based on Jensen's inequality, it is not difficult to verify that, for some $c > 0$ independent of n ,

$$\int_{\Omega} \log^- \vartheta_{0,n}^{(2)} \leq c \left(1 + \int_{\Omega} \log^- \vartheta_0 \right). \quad (5.27)$$

Finally, we pass to the boundary component. First of all, thanks to smoothness of Γ , we can find $\epsilon > 0$ such that η_0 can be extended (e.g., constantly along directions orthogonal to Γ) to a function $\eta_{0,*}$ defined on a neighbourhood $\Gamma_{\delta} := \{x \in \mathbb{R}^3 : d(x, \Gamma) \leq \delta\}$ of Γ . Thanks to Fubini's theorem, it is then clear that both $\eta_{0,*}$ and $\log \eta_{0,*}$ lie in $L^1(\Gamma_{\delta})$. Then, we truncate $\eta_{0,*}$ (as in (5.26)) obtaining $\eta_{0,n}^{(1)}$ (which can be seen as a function defined on the whole of \mathbb{R}^3). Next, we mollify $\eta_{0,n}^{(1)}$, introducing $\eta_{0,n}^{(2)} := \rho_n * \eta_{0,n}^{(1)}$, for ρ_n supported, say, on the ball $\bar{B}(0, 1/n)$. Finally, we take a cutoff function $\psi_n \in C^\infty(\mathbb{R}^3; [0, 1])$ such that ψ_n is identically one on $\Gamma_{1/2n}$ and ψ_n is supported on $\Gamma_{1/n}$. Then, we set $\eta_{0,n}^{(3)} := \eta_{0,n}^{(2)} \psi_n$ in such a way that $\eta_{0,n}^{(3)}$ belongs to $C^\infty(\mathbb{R}^3)$ and $\eta_{0,n}^{(3)}$ tends to 0 in $L^1(\Omega)$, while its trace tends to η_0 in $L^1(\Gamma)$. Moreover, as above, one can check that $\log^- (\eta_{0,n}^{(3)})$ is uniformly controlled in $L^1(\Gamma)$ in the sense of (5.27). Then, the required approximation of ϑ_0 is obtained simply taking $\vartheta_{0,n} := \vartheta_{0,n}^{(2)} + \eta_{0,n}^{(3)}$.

5.3 Solutions with regularizing effects for ϑ

In this last part, we extend the previous results in three directions. First, we prove that, if ϑ_0 (and, possibly, η_0) enjoy higher summability properties, then there exist weak solutions whose component ϑ satisfies time-regularization properties in the spirit of Lemmas 3.4 and 3.5. Second, we demonstrate that, under the same type of conditions on the initial data, existence holds also for $\alpha = 0$ and $\beta > 0$ (recall that we could not deal with this case for L^1 initial data, cf. Theorem 5.1). Third, we see that uniqueness holds in the class of solutions with regularizing effects. All these results are collected in the following

Theorem 5.4. *Let assumptions (2.10)-(2.11) and (2.13) hold and let in addition, for some $\epsilon \in (0, 1)$,*

$$f \in L^2(0, +\infty; L^{3+\epsilon}(\Omega)), \quad N_\epsilon := \|f\|_{L^2(0, +\infty; L^{3+\epsilon}(\Omega))}. \quad (5.28)$$

Moreover, if $\alpha > 0$ and $\beta = 0$, let

$$\vartheta_0 \in L^{4+\epsilon}(\Omega), \quad \alpha \vartheta_0 \in L^{4+\epsilon}(\Gamma), \quad (5.29)$$

whereas in the other cases let

$$\vartheta_0 \in L^{3+\epsilon}(\Omega), \quad \alpha \vartheta_0 \in L^{3+\epsilon}(\Gamma). \quad (5.30)$$

Then, Problem (P) admits at least one energy solution (ϑ, u) satisfying (5.1), (5.2), together with the regularization estimate

$$\|\vartheta(t)\|_{L^\infty(\Omega)} \leq Q(\mathbb{E}_{0+\epsilon}, N_\epsilon, \tau^{-1}) \quad \forall t \geq \tau, \quad \tau \in (0, 1), \quad (5.31)$$

where we have set

$$\mathbb{E}_{0+\epsilon} := \mathcal{E}(\vartheta_0) + \|\vartheta_0\|_{L^{3+\epsilon}(\Omega)}^{3+\epsilon} + \alpha \|\vartheta_0\|_{L^{3+\epsilon}(\Gamma)}^{3+\epsilon}, \quad (5.32)$$

the exponents $3 + \epsilon$ being all replaced by $4 + \epsilon$ in the case when $\alpha > 0$ and $\beta = 0$. Moreover, in all cases with the exception of $\alpha = 0$ and $\beta > 0$, we have

$$\begin{aligned} & \|\vartheta_t\|_{L^2(t, t+1; H)} + \alpha \|\eta_t\|_{L^2(t, t+1; H_\Gamma)} + \|u\|_{L^\infty(t, +\infty; V)} + \beta \|\eta\|_{L^\infty(t, \infty; V_\Gamma)} \\ & \leq Q(\mathbb{E}_{0+\epsilon}, N_\epsilon, \tau^{-1}) \quad \forall t \geq \tau, \quad \tau \in (0, 1). \end{aligned} \quad (5.33)$$

Finally, whenever there exists an energy solution satisfying (5.31) and (5.33), then it is the unique solution in that regularity class.

PROOF. For the sake of simplicity, we just prove the theorem by directly working on the “limit” solutions without referring to an explicit approximation scheme. That said, we first observe that the energy estimate (4.21) still holds. Moreover, we can still rely on the conclusion of Lemma 3.1. Next, thanks to assumption (5.28), we can apply Lemma 3.3 over the generic time interval $(0, T)$. This gives

$$\|\vartheta(t)\|_{L^{3+\epsilon}(\Omega)} + \alpha \|\vartheta(t)\|_{L^{3+\epsilon}(\Gamma)} \leq Q(N_\epsilon, \mathbb{E}_{0+\epsilon}) \quad (5.34)$$

(here and below, $3 + \epsilon$ is replaced by $4 + \epsilon$ in case $\alpha > 0$ and $\beta = 0$). Hence, for $t \geq \tau > 0$, we can apply Lemma 3.4 (or Lemma 3.5) over the generic time interval $(t, t + 1)$. This gives (5.31).

Finally, in all cases with the exception of $\alpha = 0$ and $\beta > 0$, we can apply Lemma 3.7 over the generic time interval $(t, t + 1)$ where $t \geq \tau > 0$, which gives (5.33).

Then, to conclude the proof, it just remain to show that, in the case when $\alpha = 0$ and $\beta > 0$ (that we set equal to 1 for simplicity), a weak solution still exists under the above assumptions. To this aim, we consider the system

$$\vartheta_{n,t} - \Delta u_n = f, \quad \vartheta_n = -1/u_n, \quad \text{in } \Omega, \quad (5.35)$$

$$\frac{1}{n} \eta_{n,t} - \Delta_\Gamma \eta_n = -\partial_n u, \quad \text{on } \Gamma, \quad (5.36)$$

complemented with the usual initial conditions. Then, for all $n \in \mathbb{N}$, there exists at least one weak solution (ϑ_n, u_n) . Moreover, thanks to (5.33) and to regularity arguments similar to those performed in Section 4.3, (ϑ_n, u_n) is smooth enough in order for the system to make sense in the above “strong” form, at least on time intervals of the form (τ, T) for all $\tau > 0$.

In addition to that, we still have estimates (5.1) and (5.2). Moreover, it is worth noting that (5.2) holds *independently of n* . Indeed, looking back at the proof of Lemma 3.1, it is immediate to check that estimate (3.6) is independent of α . Actually, when one performs the iteration argument (see (3.29)) one simply has a functional J_i that depends on α , but the bound for such a functional remains unchanged. The same holds when applying Lemma 3.4. In conclusion, we have the uniform bound

$$\|\vartheta_n(t)\|_{L^\infty(\Omega)} + \|u_n(t)\|_{L^\infty(\Omega)} \leq Q(\mathbb{E}_{0+\epsilon}, N_\epsilon, \tau^{-1}) \quad \forall t \geq \tau, \quad \tau \in (0, 1). \quad (5.37)$$

Thus, thanks also to Lemma 3.3, for any $\tau \in (0, 1)$, $T \geq \tau$, we have

$$\vartheta_n \rightarrow \vartheta \quad \text{weakly star in } L^\infty(0, T; L^{3+\epsilon}(\Omega)) \quad \text{and weakly star in } L^\infty((\tau, T) \times \Omega). \quad (5.38)$$

Moreover, since $\beta > 0$, as an additional consequence of estimate (3.39) (with $p = 3 + \epsilon$) we have

$$\|\vartheta_n^{\frac{1+\epsilon}{2}}\|_{L^2(0, T; V)} + \|\eta_n^{\frac{3+\epsilon}{2}}\|_{L^2(0, T; V_\Gamma)} \leq c. \quad (5.39)$$

In particular, being

$$\nabla \vartheta_n = \frac{2}{1 + \epsilon} \vartheta_n^{\frac{1-\epsilon}{2}} \nabla \vartheta_n^{\frac{1+\epsilon}{2}}, \quad (5.40)$$

we have, from (5.38)-(5.39),

$$\|\nabla \vartheta_n\|_{L^2(0,T;L^{\frac{3+\epsilon}{2}}(\Omega))} \leq c \|\nabla \vartheta_n^{\frac{1+\epsilon}{2}}\|_{L^2(0,T;H)} \|\vartheta_n^{\frac{1-\epsilon}{2}}\|_{L^\infty(0,T;L^{\frac{6+2\epsilon}{1-\epsilon}}(\Omega))} \leq c. \quad (5.41)$$

In particular, this fact tells us that, in the present regularity setting, η_n can be directly seen as the trace of ϑ_n . More precisely, applying the trace theorem, we have

$$\|\eta_n\|_{L^2(0,T;W^{\frac{1+\epsilon}{3+\epsilon},\frac{3+\epsilon}{2}}(\Gamma))} \leq c. \quad (5.42)$$

On the other hand, testing (3.2) by a generic function $\phi \in H_0^1(\Omega)$ of unit norm and recalling (5.1) and (5.28), we obtain

$$\vartheta_{n,t} \rightarrow \vartheta_t \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)). \quad (5.43)$$

Thus, using (5.41), (5.43) and the Aubin-Lions lemma, we infer

$$\vartheta_n \rightarrow \vartheta \quad \text{strongly in } L^2(0,T;W^{1-\sigma,\frac{3+\epsilon}{2}}(\Omega)) \quad (5.44)$$

for all $\sigma \in (0,1)$. Hence, by continuity of the trace operator,

$$\eta_n \rightarrow \eta = \vartheta|_\Gamma \quad \text{strongly in } L^2(0,T;L^p(\Gamma)) \quad \text{for some } p > 1. \quad (5.45)$$

Thus, writing the weak formulation (2.19) of system (5.35)-(5.36), it is immediate to check that one can take the limit $n \rightarrow \infty$ therein. In particular, it can be standardly proved that

$$u_n \rightarrow u = -1/\vartheta \quad \text{strongly in } L^p(0,T;L^p(\Omega)), \quad \text{say, for all } p \in [1,2). \quad (5.46)$$

Hence, in place of (2.17), we can now directly write $\eta = \vartheta|_\Gamma$ in the sense of traces and a.e. in $(0,T)$. Moreover, thanks to the additional regularity properties coming from (5.41)-(5.42), we could even relax a bit the requirements (2.18) on the test function ξ (we omit the details).

Finally, we come to uniqueness. Let us given a couple of energy solutions (ϑ_1, u_1) and (ϑ_2, u_2) both satisfying (5.31) and (5.33). Thanks to smoothness for strictly positive times, we can proceed as in Section 5.1 testing the difference of the equations by $\text{sign}_\varepsilon(u_1 - u_2)$. We then integrate over (τ, T) for $\tau > 0$ and arrive at the analogue of (5.13), namely

$$\begin{aligned} & \int_\tau^t ((\vartheta_{1,t} - \vartheta_{2,t}), \text{sign}_\varepsilon(u_1 - u_2)) + \alpha \int_\tau^t ((\eta_{1,t} - \eta_{2,t}), \text{sign}_\varepsilon(u_1 - u_2))_\Gamma \\ & - \beta \int_\tau^t (\Delta_\Gamma(\eta_1 - \eta_2), \text{sign}_\varepsilon(u_1 - u_2))_\Gamma \leq 0. \end{aligned} \quad (5.47)$$

Note that we cannot integrate directly over $(0,t)$ since (5.31) and (5.33) do not extend to $\tau = 0$; in other words, we do not have sufficient regularity to use $\text{sign}_\varepsilon(u_1 - u_2)$ as a test function over $(0,T)$. Hence, we first need to take the limit $\varepsilon \searrow 0$, obtaining

$$\|\vartheta_1(t) - \vartheta_2(t)\|_1 + \alpha \|\eta_1(t) - \eta_2(t)\|_{1,\Gamma} \leq \|\vartheta_1(\tau) - \vartheta_2(\tau)\|_1 + \alpha \|\eta_1(\tau) - \eta_2(\tau)\|_{1,\Gamma}. \quad (5.48)$$

Letting $\tau \searrow 0$ and noting that energy solutions are continuous with values in L^1 (cf. (2.14)), we obtain the assert. The proof is concluded. \blacksquare

Remark 5.5. In principle, the uniqueness proof works also in case (5.30) (or (5.29)) is not satisfied, i.e., under the conditions of Theorem 5.1. However, in that case the statement may be vacuous since we do not know whether there exist energy solutions satisfying (5.31) and (5.33). Actually, even in case (5.30) (or (5.29)) holds, the proved properties does not exclude that there might exist other energy solutions of Problem (P) that *do not* regularize with respect to time. The same observation can also be referred to the case when $\alpha = 0$ and $\beta > 0$ since (5.33) is not known to hold under these conditions.

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